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Symmetry Techniques and Generating Functions for Basic Analogue of Fox's H-Function

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ABSTRACT

The objective of this paper is to establish the q-recurrence relations, q-difference equations for basic analogue of Fox's H-function and then obtain the canonical equation associated with each family of multivariable q-analogue of Fox's H-function and generating functions of families of basic analogue of Fox's H-function. A significantly large number of works on the subject of H- function gives interesting account of the theory and its applications in many different areas of mathematical analysis. A lot of research work has been recently come up on the study and development of a function that is more general.

1.1 Present paper deals with symmetry techniques for derivation of generating functions of families of basic analogue of Fox's H-function. To each family of basic analogue of Fox's H-function, a canonical system of partial q-difference equations has been associated. Subsequently symmetries of these equations have been used to derive the generating functions.

First we derive the q-recurrence relations, q-difference equations for basic analogue of Fox's H-function and then obtain the canonical equation associated with each family of multivariable q-analogue of Fox's H-function. Also, we can derive the generating function for symmetry operators.

Saxena, et. al. [9] introduced the basic analogue of the H-function in terms of the Mellin Barnes type basic contour integral in the following form:

$$H_{A,B}^{m,n}\left[z;q\middle|\begin{matrix}(a_1,\alpha_1)\ldots\ldots\ldots\ldots(a_A,\alpha_A)\\(b_1,\beta_1)\ldots\ldots\ldots\ldots(b_B,\beta_B)\end{matrix}\right]$$

$$=\frac{1}{2\pi i}\int_C\frac{\prod_{j=1}^{m}G\left(q^{b_j-\beta_j s}\right)\prod_{j=1}^{n}G\left(q^{1-a_j+\alpha_j s}\right)\pi z^s}{\prod_{j=m+1}^{B}G\left(q^{1-b_j+\beta_j s}\right)\prod_{j=n+1}^{A}G\left(q^{a_j-\alpha_j s}\right)G\left(q^{1-s}\right)\sin\pi s} \quad ....(1.1.1)$$

Where,

$$G(q\alpha)=\left\{\prod_{n=0}^{\infty}\left(1-q^{\alpha+n}\right)\right\}^{-1}=\frac{1}{\left(q^{\alpha};q\right)_{\infty}} \quad ....(1.1.2)$$

and $0 \le m \le B$; $0 \le n \le A$; αj, βj are all positive integers. The contour C is a line parallel to Re(ws) = 0 with indentations, if necessary, in such a manner that all the poles of $G(q^{bj-\beta js})$; $1 \le j \le n$ are to its left of C. The integral converges if Re [s log(z) – log sin πs] < 0 for large value of |s| on the contour C, that is, if $|\{\arg(z) - w_2 w_1^{-1} \log |z|\}| < \pi$ where |q|<1, log $q = -w = w(w_1+iw_2)$, w, w_1,w_2 are definite quantities, w_1 and w_2 being real. The Fox's H-function has been studied in detail by several mathematicians for its theoretical ad applications points of view. This function has found wide-ranging applications in mathematical, physical, biological and statistical sciences. It would be interesting to observe that almost all the classical special functions expressible in terms of Fox's H-function along with their applications to the aforementioned fields can be found in the research monograph by Mathai, et. al. [3,4].

A new generalization was considered by Saxena, et. al. [9] in the form of the q-extensions of the Fox's H-function by mean of the Mellin – Barne's type of basic integral. The advantage of these new extensions of the Fox's H-functions lies in the fact that a number of q-special functions including the basic hyper geometric functions, happen to be the particular cases of the Hq(.)-functions, thus widening the scope for further applications. In a paper, Saxena, et. al. [8] besides proving some interesting relations, have established an important limit formula for the Hq(.)-function, when q tends to 1. Various basic functions expressible in terms of basic analogue of Fox's H-function with their applications can be found in the research papers due to Saxena, et. al. [10], Yadav, et. al. [11] and Purohit, et. al. [7].

1.2 Recurrence Relations

The following recurrence relations will be established in this section.

$$\text{(i)}\ \left(1-q^{1-a_j}T_z^{\alpha_j}\right)H_{A,B}^{m,n}\left[z;q\middle|\begin{matrix}(a_1,\alpha_1)\ldots\ldots\ldots,(a_A,\alpha_A)\\(b_1,\beta_1)\ldots\ldots\ldots,(b_B,\beta_B)\end{matrix}\right]$$

$$=H_{A,B}^{m,n}\left[z;q\middle|\begin{matrix}(a_1,\alpha_1)\ldots,(a_j-1,\alpha_j)\ldots,(a_n,\alpha_n),(a_{n+1},\alpha_{n+1})\ldots\ldots,(a_A,\alpha_A)\\(b_1,\beta_1),\qquad\ldots\ldots\ldots,\qquad(b_B,\beta_B)\end{matrix}\right]$$

$$1 \le j \le n \quad \text{....(1.2.1)}$$

$$\text{(ii)}\ \left(1-q^{a_j-1}T_z^{-\alpha_j}\right)H_{A,B}^{m,n}\left[z;q\middle|\begin{matrix}(a_1,\alpha_1)\ldots\ldots\ldots,(a_A,\alpha_A)\\(b_1,\beta_1)\ldots\ldots\ldots,(b_B,\beta_B)\end{matrix}\right]$$

$$=H_{A,B}^{m,n}\left[z;q\middle|\begin{matrix}(a_1,\alpha_1)\ldots,(a_n,\alpha_n),(a_{n+1},\alpha_{n+1}),\ldots\ldots(a_j-1,\alpha_j),,(a_A,\alpha_A)\\(b_1,\beta_1),\qquad\ldots\ldots\ldots,\qquad(b_B,\beta_B)\end{matrix}\right]$$

$$n+1 \le j \le A \quad \text{....(1.2.2)}$$

$$\text{(iii)}\ \left(1-q^{b_j}T_z^{-\beta_j}\right)H_{A,B}^{m,n}\left[z;q\middle|\begin{matrix}(a_1,\alpha_1)\ldots\ldots\ldots,(a_A,\alpha_A)\\(b_1,\beta_1)\ldots\ldots\ldots,(b_B,\beta_B)\end{matrix}\right]$$

$$=H_{A,B}^{m,n}\left[z;q\middle|\begin{matrix}(a_1,\alpha_1),\qquad\ldots\ldots\ldots\ldots\ldots,\qquad(a_A,\alpha_A)\\(b_1,\beta_1)\ldots,(b_l+1,\beta_j)\ldots,(b_m,\beta_m),(b_{m+1},\beta_{m+1}),\ldots\ldots(b_1,\beta_B)\end{matrix}\right]$$

$1 \le j \le m$ ....(1.2.3)

(iv) $$\left(1-q^{-b_j}T_z^{\beta_j}\right)H_{A,B}^{m,n}\left[z;q\left|\begin{matrix}(a_1,\alpha_1)........,(a_A,\alpha_A)\\(b_1,\beta_1).........,(b_B,\beta_B)\end{matrix}\right.\right]$$

$$=H_{A,B}^{m,n}\left[z;q\left|\begin{matrix}(a_1,\alpha_1), & .... & (a_A,\alpha_A)\\(b_1,\beta_1)...,(b_m,\beta_m),(b_{m+1},\beta_{m+1}),.....(b_l+1,\beta_j).....,(b_1,\beta_B)\end{matrix}\right.\right]$$

$1 \le j \le m$ ....(1.2.4)

(v) $$z\left(1-T_z^{-1}\right)H_{A,B}^{m,n}\left[z;q\left|\begin{matrix}(a_1,\alpha_1)........,(a_A,\alpha_A)\\(b_1,\beta_1).........,(b_B,\beta_B)\end{matrix}\right.\right]$$

$$=-H_{A,B}^{m,n}\left[z;q\left|\begin{matrix}(a_1+\alpha_1,\alpha_1),...........(a_A+\alpha_A,\alpha_A)\\(b_1+\beta_1,\beta_1),.............,(b_B+\beta_B,\beta_B)\end{matrix}\right.\right] \quad ....(1.2.5)$$

Proof of (i) To prove the results, we consider

L.H.S. = $$\left(1-q^{1-a_j}T_z^{\alpha_j}\right)H_{A,B}^{m,n}\left[z;q\left|\begin{matrix}(a_1,\alpha_1)........,(a_A,\alpha_A)\\(b_1,\beta_1).........,(b_B,\beta_B)\end{matrix}\right.\right],$$

$1 \le j \le n$

On making use of definition (1.1.1) and q-dilation operator defined by Miller, et. al. [1]. The above expression becomes

$$\frac{1}{2\pi i}\int_C \frac{\prod_{j=1}^{m} G\left(q^{b_j-\beta_j s}\right)\prod_{j=1}^{n} G\left(q^{1-a_j+\alpha_j s}\right)\left(1-q^{1-a_j+\alpha_j s}\right)\pi z^s}{\prod_{j=m+1}^{B} G\left(q^{1-b_j+\beta_j s}\right)\prod_{j=n+1}^{A} G\left(q^{a_j-\alpha_j s}\right)G\left(q^{1-s}\right)\sin \pi s}$$

by using (1.1.2), it becomes

$$H_{A,B}^{m,n}\left[z;q\left|\begin{matrix}(a_1,\alpha_1)...,(a_j-1,\alpha_j),.....,(a_n,\alpha_n),(a_{n+1},\alpha_{n+1}),......(a_A,\alpha_A)\\(b_1,\beta_1), \quad ...., \quad ,(b_1,\beta_B)\end{matrix}\right.\right]$$

$1 \le j \le n$

= R.H.S.

This completes the proof.

Proof of (ii) To prove the result,

we consider

L.H.S. = $\left(1-q^{a_j-1}T_z^{-\alpha_j}\right)H_{A,B}^{m,n}\left[z;q\left|\begin{matrix}(a_1,\alpha_1)........,(a_A,\alpha_A)\\(b_1,\beta_1)........,(b_B,\beta_B)\end{matrix}\right.\right]$,

On making use of definition (1.1.1) and q-dilation operator, the above expression becomes

$$\frac{1}{2\pi i}\int_C \frac{\prod_{j=1}^{m}G\left(q^{b_j-\beta_j s}\right)\prod_{j=1}^{n}G\left(q^{1-a_j+\alpha_j s}\right)\left(1-q^{1-a_j-\alpha_j s}\right)\pi z^s}{\prod_{j=m+1}^{B}G\left(q^{1-b_j+\beta_j s}\right)\prod_{j=n+1}^{A}G\left(q^{a_j-\alpha_j s}\right)G\left(q^{1-s}\right)\sin \pi s}$$

by using (1.1.2), it becomes

$$H_{A,B}^{m,n}\left[z;q\left|\begin{matrix}(a_1,\alpha_1).......,(a_n,\alpha_n),(a_{n+1},\alpha_{n+1}),...(a_j-1,\alpha_j),...,(a_A,\alpha_A)\\(b_1,\beta_1), \qquad ....,\qquad ,(b_1,\beta_B)\end{matrix}\right.\right]$$

$;n+1\le j\le A$

= R.H.S.

This completes the proof.

Proof of (iii) To prove the result,

We consider

L.H.S. = $\left(1-q^{b_j}T_z^{-\beta_j}\right)H_{A,B}^{m,n}\left[z;q\left|\begin{matrix}(a_1,\alpha_1)........,(a_A,\alpha_A)\\(b_1,\beta_1)........,(b_B,\beta_B)\end{matrix}\right.\right]$,

On making use of definition (1.1.1) and q-dilation operator, the above expression becomes

$$\frac{1}{2\pi i}\int_C \frac{\prod_{j=1}^{m}G\left(q^{b_j-\beta_j s}\right)\left(1-q^{b_j-\beta_j s}\right)\prod_{j=1}^{n}G\left(q^{1-a_j+\alpha_j s}\right)\pi z^s}{\prod_{j=m+1}^{B}G\left(q^{1-b_j+\beta_j s}\right)\prod_{j=n+1}^{A}G\left(q^{a_j-\alpha_j s}\right)G\left(q^{1-s}\right)\sin \pi s}$$

by using (1.1.2), it becomes

$$=H_{A,B}^{m,n}\left[z;q\left|\begin{matrix}(a_1,\alpha_1), \qquad .... \qquad (a_A,\alpha_A)\\(b_1,\beta_1)...,(b_l+1,\beta_j)....,(b_m,\beta_m),(b_{m+1},\beta_{m+1}),.......(b_1,\beta_B)\end{matrix}\right.\right]$$

$;1\le j\le m$

= R.H.S.

This completes the proof.

Proof of (iv) To prove the result, we consider

L.H.S. = $\left(1-q^{-b_j}T_z^{\beta_j}\right)H_{A,B}^{m,n}\left[z;q\left|\begin{matrix}(a_1,\alpha_1)........,(a_A,\alpha_A)\\(b_1,\beta_1)........,(b_B,\beta_B)\end{matrix}\right.\right],$

$$m+1\le j\le B$$

On making use of definition (1.1.1) and q-dilation operator, the above expression becomes

$$\frac{1}{2\pi i}\int_C\frac{\prod_{j=1}^{m}G\left(q^{b_j-\beta_j s}\right)\left(1-q^{b_j+\beta_j s}\right)\prod_{j=1}^{n}G\left(q^{1-a_j+\alpha_j s}\right)\pi z^s}{\prod_{j=m+1}^{B}G\left(q^{1-b_j+\beta_j s}\right)\prod_{j=n+1}^{A}G\left(q^{a_j-\alpha_j s}\right)G\left(q^{1-s}\right)\sin\pi s}$$

by using (1.1.2), it becomes

$$=H_{A,B}^{m,n}\left[z;q\left|\begin{matrix}(a_1,\alpha_1), & .... & (a_A,\alpha_A)\\(b_1,\beta_1).......,(b_m,\beta_m),(b_{m+1},\beta_{m+1}),.....(b_l+1,\beta_j)......(b_1,\beta_B)\end{matrix}\right.\right]$$

$$;m+1\le j\le B$$

= R.H.S.

This completes the proof.

Proof of (v) To prove the result, we consider

L.H.S. = $z\left(1-T_z^{-1}\right)H_{A,B}^{m,n}\left[z;q\left|\begin{matrix}(a_1,\alpha_1)........,(a_A,\alpha_A)\\(b_1,\beta_1)........,(b_B,\beta_B)\end{matrix}\right.\right],$

On making use of definition (1.1.1) and q-dilation operator, the above expression becomes

$$\frac{1}{2\pi i}\int_C\frac{\prod_{j=1}^{m}G\left(q^{b_j-\beta_j s}\right)\prod_{j=1}^{n}G\left(q^{1-a_j+\alpha_j s}\right)\left(1-q^{-s}\right)\pi z^{s+1}}{\prod_{j=m+1}^{B}G\left(q^{1-b_j+\beta_j s}\right)\prod_{j=n+1}^{A}G\left(q^{a_j-\alpha_j s}\right)G\left(q^{1-s}\right)\sin\pi s}$$

This integral takes the form

$$\frac{1}{2\pi i}\int_C\frac{\prod_{j=1}^{m}G\left(q^{b_j-\beta_j s}\right)\prod_{j=1}^{n}G\left(q^{1-a_j+\alpha_j s}\right)\pi z^{s+1}}{\prod_{j=m+1}^{B}G\left(q^{1-b_j+\beta_j s}\right)\prod_{j=n+1}^{A}G\left(q^{a_j-\alpha_j s}\right)G\left(q^{-s}\right)\sin\pi s}ds;\ \text{using (1.1.2)}$$

Replacing s+1 by t in this integral, it becomes

$$-\frac{1}{2\pi i}\int_C \frac{\prod_{j=1}^{m} G\left(q^{(b_j-\beta_j s)-\beta_j t}\right)\prod_{j=1}^{n} G\left(q^{1-(a_j+\alpha_j)+\alpha_j t}\right)\pi z^t}{\prod_{j=m+1}^{B} G\left(q^{(1-b_j+\beta_j s)+\beta_j t}\right)\prod_{j=n+1}^{A} G\left(q^{(a_j+\alpha_j)-\alpha_j t}\right)G\left(q^{1-y}\right)\sin \pi t} dt$$

which equals

$$-H_{A,B}^{m,n}\left[z;q\middle|\begin{matrix}(a_1+\alpha_1,\alpha_1),\ldots\ldots\ldots\ldots(a_A+\alpha_A,\alpha_A)\\(b_1+\beta_1,\beta_1),\ldots\ldots\ldots\ldots,(b_B+\beta_B,\beta_B)\end{matrix}\right]$$

= R.H.S.

This completes the proof.

Note that, relations (1.2.1) to (1.2.5) imply the fundamental q-difference equation satisfied by $H_{A,B}^{m,n}$.

$$\{z\left(1-T_z^{-1}\right)\left(1-q^{1-a_1}T_z^{\alpha_1}\right)^{\alpha_1}\ldots\left(1-q^{1-a_n}T_z^{\alpha_n}\right)^{\alpha_n}\left(1-q^{a_{n+1}-1}T_z^{-\alpha_{n+1}}\right)^{\alpha_{n+1}}$$

$$\ldots\left(1-q^{a_A-1}T_z^{-\alpha_A}\right)^{\alpha_A}+\left(1-q^{b_1}T_z^{-\beta_1}\right)^{\beta_1}\ldots\left(1-q^{b_m}T_z^{-\beta_m}\right)^{\beta_m}$$

$$\left(1-q^{-b_{m+1}}T_z^{\beta_{m+1}}\right)^{\beta_{m+1}}\ldots\left(1-q^{-b_B}T_z^{\beta_B}\right)^{\beta_B}\}H_{A,B}^{m,n}=0$$

......(1.2.6)

1.3 Canonical Equations

Now, we define the basis function $\psi_{A,B}^{m,n}$ of A+B+1 variables by

$$\psi_{A,B}^{m,n}\left[\frac{u_{A+1}^{\beta_1}\ldots\ldots u_{A+B}^{\beta_B}}{u_1^{\alpha_1}\ldots\ldots u_A^{\alpha_A}}\cdot\frac{1}{u_{A+B+1}};q\middle|\begin{matrix}(a_1,\alpha_1)\ldots\ldots\ldots,(a_A,\alpha_A)\\(b_1,\beta_1)\ldots\ldots\ldots,\ (b_B,\beta_B)\end{matrix}\right]$$

$$=H_{A,B}^{m,n}\left[\frac{u_{A+1}^{\beta_1}\ldots\ldots u_{A+B}^{\beta_B}}{u_1^{\alpha_1}\ldots\ldots u_A^{\alpha_A}}\cdot\frac{1}{u_{A+B+1}};q\middle|\begin{matrix}(a_1,\alpha_1)\ldots\ldots\ldots,(a_A,\alpha_A)\\(b_1,\beta_1)\ldots\ldots\ldots,\ (b_B,\beta_B)\end{matrix}\right]$$

$$u_1^{a_1-1}\ldots u_A^{a_A-1}u_{A+1}^{-b_1}\ldots u_{A+B}^{-b_B}$$ ......(1.3.1)

The q-difference operators $\Delta_j^{\pm}$ are defined by

$$\Delta_j^{\pm} f\left(u_j\right)=u_j^{-1}[f(u_j)-f(q^{\pm}u_j)]$$ .....(1.3.2)

In terms of operator (1.3.2) and with the help of the relations (1.2.1) to (1.2.5), we obtain the following recurrence relations.

(i) $\Delta_j^{-}\psi_{A,B}^{m,n}\left[\frac{u_{A+1}^{\beta_1}\ldots\ldots u_{A+B}^{\beta_B}}{u_1^{\alpha_1}\ldots\ldots u_A^{\alpha_A}}\cdot\frac{1}{u_{A+B+1}};q\left|\begin{matrix}(a_1,\alpha_1)\ldots\ldots\ldots,(a_A,\alpha_A)\\(b_1,\beta_1)\ldots\ldots\ldots,\ (b_B,\beta_B)\end{matrix}\right.\right]$

$-\psi_{A,B}^{m,n}\left[\frac{u_{A+1}^{\beta_1}\ldots\ldots u_{A+B}^{\beta_B}}{u_1^{\alpha_1}\ldots\ldots u_A^{\alpha_A}}\cdot\frac{1}{u_{A+B+1}};\right.$

$q\left|\begin{matrix}(a_1,\alpha_1)\ldots\ldots,(a_j-1,\alpha_j)\ldots,(a_n,\alpha_n),(a_{n+1},\alpha_{n+1})\ldots,(a_A,\alpha_A)\\(b_1,\beta_1), \qquad \ldots\ldots\ldots, \qquad (b_B,\beta_B)\end{matrix}\right.$

$1\leq j\leq n$ ..... (1.3.3)

(ii) $\Delta_j^{-}\psi_{A,B}^{m,n}\left[\frac{u_{A+1}^{\beta_1}\ldots\ldots u_{A+B}^{\beta_B}}{u_1^{\alpha_1}\ldots\ldots u_A^{\alpha_A}}\cdot\frac{1}{u_{A+B+1}};q\left|\begin{matrix}(a_1,\alpha_1)\ldots\ldots\ldots,(a_A,\alpha_A)\\(b_1,\beta_1)\ldots\ldots\ldots,\ (b_B,\beta_B)\end{matrix}\right.\right]$

$-\psi_{A,B}^{m,n}\left[\frac{u_{A+1}^{\beta_1}\ldots\ldots u_{A+B}^{\beta_B}}{u_1^{\alpha_1}\ldots\ldots u_A^{\alpha_A}}\cdot\frac{1}{u_{A+B+1}};\right.$

$q\left|\begin{matrix}(a_1,\alpha_1)\ldots\ldots\ldots,(a_n,\alpha_n),(a_{n+1},\alpha_{n+1})(a_j-1,\alpha_j)\ldots,(a_A,\alpha_A)\\(b_1,\beta_1), \qquad \ldots\ldots\ldots, \qquad (b_B,\beta_B)\end{matrix}\right.$

; $n+1\leq j\leq A$ ..... (1.3.4)

(iii) $\Delta_j^{-}\psi_{A,B}^{m,n}\left[\frac{u_{A+1}^{\beta_1}\ldots\ldots u_{A+B}^{\beta_B}}{u_1^{\alpha_1}\ldots\ldots u_A^{\alpha_A}}\cdot\frac{1}{u_{A+B+1}};q\left|\begin{matrix}(a_1,\alpha_1)\ldots\ldots\ldots,(a_A,\alpha_A)\\(b_1,\beta_1)\ldots\ldots\ldots,\ (b_B,\beta_B)\end{matrix}\right.\right]$

$-\psi_{A,B}^{m,n}\left[\frac{u_{A+1}^{\beta_1}\ldots\ldots u_{A+B}^{\beta_B}}{u_1^{\alpha_1}\ldots\ldots u_A^{\alpha_A}}\cdot\frac{1}{u_{A+B+1}};\right.$

$q\left|\begin{matrix}(a_1,\alpha_1), \qquad \ldots\ldots\ldots\ldots\ldots \qquad (a_A,\alpha_A)\\(b_1,\beta_1)\ldots\ldots,(b_l+1,\beta_j)\ldots(b_m,\beta_m),(b_{m+1},\beta_{m+1}),\ldots\ldots(b_B,\beta_B)\end{matrix}\right.$

; $A+1\leq j\leq A+m$ ..... (1.3.5)

(iv) $\Delta_j^{+}\psi_{A,B}^{m,n}\left[\frac{u_{A+1}^{\beta_1}\ldots\ldots u_{A+B}^{\beta_B}}{u_1^{\alpha_1}\ldots\ldots u_A^{\alpha_A}}\cdot\frac{1}{u_{A+B+1}};q\left|\begin{matrix}(a_1,\alpha_1)\ldots\ldots\ldots,(a_A,\alpha_A)\\(b_1,\beta_1)\ldots\ldots\ldots,\ (b_B,\beta_B)\end{matrix}\right.\right]$

$-\psi_{A,B}^{m,n}\left[\frac{u_{A+1}^{\beta_1}\ldots\ldots u_{A+B}^{\beta_B}}{u_1^{\alpha_1}\ldots\ldots u_A^{\alpha_A}}\cdot\frac{1}{u_{A+B+1}};\right.$

$$q\left|\begin{matrix}(a_1,\alpha_1), & \ldots\ldots\ldots\ldots\ldots\ldots & (a_A,\alpha_A)\\ (b_1,\beta_1)\ldots\ldots,(b_m,\beta_m),(b_{m+1},\beta_{m+1}),\ldots,(b_l+1,\beta_j)\ldots,(b_B,\beta_B)\end{matrix}\right.$$

$$A+m+1\le j\le A+B \;\ldots..\;(1.3.6)$$

(v) $\Delta^{+}_{A+B+1}\psi^{m,n}_{A,B}\left[\dfrac{u_{A+1}^{\beta_1}\ldots\ldots u_{A+B}^{\beta_B}}{u_1^{\alpha_1}\ldots\ldots u_A^{\alpha_A}}.\dfrac{1}{u_{A+B+1}};q\left|\begin{matrix}(a_1,\alpha_1)\ldots\ldots\ldots,(a_A,\alpha_A)\\(b_1,\beta_1)\ldots\ldots\ldots,\ (b_B,\beta_B)\end{matrix}\right.\right]$

$$=-\psi^{m,n}_{A,B}\left[\frac{u_{A+1}^{\beta_1}\ldots\ldots u_{A+B}^{\beta_B}}{u_1^{\alpha_1}\ldots\ldots u_A^{\alpha_A}}.\frac{1}{u_{A+B+1}};\right.$$

$$q\left|\begin{matrix}(a_1+\alpha_1,\alpha_1),\ldots\ldots\ldots\ldots(a_A+\alpha_A,\alpha_A)\\(b_1+\beta_1,\beta_1),\ldots\ldots\ldots\ldots,(b_B+\beta_B,\beta_B)\end{matrix}\right. \qquad \ldots..(1.3.7)$$

Proof of (i) We consider, L.H.S

$$\Delta^{-}_{A+B+1}\psi^{m,n}_{A,B}\left[\frac{u_{A+1}^{\beta_1}\ldots\ldots u_{A+B}^{\beta_B}}{u_1^{\alpha_1}\ldots\ldots u_A^{\alpha_A}}.\frac{1}{u_{A+B+1}};q\left|\begin{matrix}(a_1,\alpha_1)\ldots\ldots\ldots,(a_A,\alpha_A)\\(b_1,\beta_1)\ldots\ldots\ldots,\ (b_B,\beta_B)\end{matrix}\right.\right],\ 1\le j\le n$$

$$=\Delta^{-}_{j}H^{m,n}_{A,B}\left[\frac{u_{A+1}^{\beta_1}\ldots\ldots u_{A+B}^{\beta_B}}{u_1^{\alpha_1}\ldots\ldots u_A^{\alpha_A}}.\frac{1}{u_{A+B+1}};q\left|\begin{matrix}(a_1,\alpha_1)\ldots\ldots\ldots,(a_A,\alpha_A)\\(b_1,\beta_1)\ldots\ldots\ldots,\ (b_B,\beta_B)\end{matrix}\right.\right]$$

$$u_1^{a_1-1}\ldots u_A^{a_A-1}u_{A+1}^{-b_1}\ldots u_{A+B}^{-b_B}$$

In view of (1.1.1) and (1.3.2) R.H.S can be expressed as

$$\frac{1}{2\pi i}\int_C\frac{\prod_{j=1}^{m}G\left(q^{b_j-\beta_j s}\right)\prod_{j=1}^{n}G\left(q^{1-a_j+\alpha_j s}\right)\left(1-q^{1-a_j+\alpha_j s}\right)\pi\left(\dfrac{u_{A+1}^{\beta_1}\ldots\ldots u_{A+B}^{\beta_B}}{u_1^{\alpha_1}\ldots\ldots u_A^{\alpha_A}}.\dfrac{1}{u_{A+B+1}}\right)^s}{\prod_{j=m+1}^{B}G\left(q^{1-b_j+\beta_j s}\right)\prod_{j=n+1}^{A}G\left(q^{a_j-\alpha_j s}\right)G\left(q^{1-s}\right)\sin\pi s}ds$$

$$u_1^{a_1-1}\ldots u_j^{(a_j-1)-1},\ldots u_n^{a_n-1}u_n^{a_{n+1}-1}u_A^{a_A-1}u_{A+1}^{-b_1}\ldots u_{A+B}^{-b_B}$$

In view of (1.1.2), it takes the form

$$H^{m,n}_{A,B}\left[\frac{u_{A+1}^{\beta_1}\ldots\ldots u_{A+B}^{\beta_B}}{u_1^{\alpha_1}\ldots\ldots u_A^{\alpha_A}}.\frac{1}{u_{A+B+1}};\right.$$

$$q\left|\begin{matrix}(a_1,\alpha_1)\ldots\ldots\ldots,(a_j-1,\alpha_j),\ldots,(a_n,\alpha_n),(a_{n+1},\alpha_{n+1})\ldots\ldots,(a_A,\alpha_A)\\(b_1,\beta_1), & \ldots\ldots\ldots, & (b_B,\beta_B)\end{matrix}\right.$$

$$u_1^{a_1-1}\ldots u_j^{(a_j-1)-1},\ldots u_n^{a_n-1}u_n^{a_{n+1}-1}u_A^{a_A-1}u_{A+1}^{-b_1}\ldots u_{A+B}^{-b_B}$$

With the help of (1.3.1), it becomes

$$\psi_{A,B}^{m,n}\left[\frac{u_{A+1}^{\beta_1}.......u_{A+B}^{\beta_B}}{u_1^{\alpha_1}......u_A^{\alpha_A}}.\frac{1}{u_{A+B+1}};\right.$$

$$q\left|\begin{matrix}(a_1,\alpha_1)........., (a_j-1,\alpha_j),....., (a_n,\alpha_n), (a_{n+1},\alpha_{n+1})......, (a_A,\alpha_A) \\ (b_1,\beta_1), \qquad ...., \qquad (b_B,\beta_B)\end{matrix}\right.$$

$$1\le j\le n$$

= R.H.S.

This completes the proof.

Proof of (ii) We consider, L.H.S.

$$\Delta_j^{+}\psi_{A,B}^{m,n}\left[\frac{u_{A+1}^{\beta_1}.......u_{A+B}^{\beta_B}}{u_1^{\alpha_1}......u_A^{\alpha_A}}.\frac{1}{u_{A+B+1}};q\left|\begin{matrix}(a_1,\alpha_1).........,(a_A,\alpha_A)\\(b_1,\beta_1)........., \ (b_B,\beta_B)\end{matrix}\right.\right],$$

$$n+1\le j\le A$$

$$=\Delta_j^{+}H_{A,B}^{m,n}\left[\frac{u_{A+1}^{\beta_1}.......u_{A+B}^{\beta_B}}{u_1^{\alpha_1}......u_A^{\alpha_A}}.\frac{1}{u_{A+B+1}};q\left|\begin{matrix}(a_1,\alpha_1).........,(a_A,\alpha_A)\\(b_1,\beta_1)........., \ (b_B,\beta_B)\end{matrix}\right.\right]$$

$$u_1^{a_1-1}....u_A^{a_A-1}u_{A+1}^{-b_1}....u_{A+B}^{-b_B}$$

In view of (1.1.1) and (1.3.2) this can be expressed as

$$\frac{1}{2\pi i}\int_C\frac{\prod_{j=1}^{m}G\left(q^{b_j-\beta_j s}\right)\prod_{j=1}^{n}G\left(q^{1-a_j+\alpha_j s}\right)\left(1-q^{1-a_j+\alpha_j s}\right)\pi\left(\frac{u_{A+1}^{\beta_1}.......u_{A+B}^{\beta_B}}{u_1^{\alpha_1}......u_A^{\alpha_A}}.\frac{1}{u_{A+B+1}}\right)^s}{\prod_{j=m+1}^{B}G\left(q^{1-b_j+\beta_j s}\right)\prod_{j=n+1}^{A}G\left(q^{a_j-\alpha_j s}\right)G\left(q^{1-s}\right)\sin\pi s}ds$$

$$u_1^{a_1-1}........u_n^{a_n-1}u_n^{a_{n+1}-1}.....u_j^{(a_j-1)-1}....u_A^{a_A-1}u_{A+1}^{-b_1}....u_{A+B}^{-b_B}$$

In view of (1.1.2), it takes the form

$$H_{A,B}^{m,n}\left[\frac{u_{A+1}^{\beta_1}.......u_{A+B}^{\beta_B}}{u_1^{\alpha_1}......u_A^{\alpha_A}}.\frac{1}{u_{A+B+1}};\right.$$

$$q\left|\begin{matrix}(a_1,\alpha_1)....., (a_n,\alpha_n), (a_{n+1},\alpha_{n+1})....., (a_j-1,\alpha_j),......., (a_A,\alpha_A) \\ (b_1,\beta_1), \qquad ...., \qquad (b_B,\beta_B)\end{matrix}\right.$$

$$u_1^{a_1-1}....u_n^{a_n-1}u_n^{a_{n+1}-1}....u_j^{(a_j-1)-1}......u_A^{a_A-1}u_{A+1}^{-b_1}....u_{A+B}^{-b_B}$$

With the help of (1.3.1), it becomes

$$\Psi_{A,B}^{m,n}\left[\frac{u_{A+1}^{\beta_1}.......u_{A+B}^{\beta_B}}{u_1^{\alpha_1}......u_A^{\alpha_A}}.\frac{1}{u_{A+B+1}};\right.$$

$$q\left|\begin{matrix}(a_1,\alpha_1).....,(a_n,\alpha_n),(a_{n+1},\alpha_{n+1})...,(a_j-1,\alpha_j),......,(a_A,\alpha_A)\\(b_1,\beta_1), \quad ...., \quad (b_B,\beta_B)\end{matrix}\right.$$

;n+1 $\le j \le A$

= R.H.S.

This completes the proof.

Proof of (iii) We consider, L.H.S.

$$\Delta_j^{-}\Psi_{A,B}^{m,n}\left[\frac{u_{A+1}^{\beta_1}.......u_{A+B}^{\beta_B}}{u_1^{\alpha_1}......u_A^{\alpha_A}}.\frac{1}{u_{A+B+1}};q\left|\begin{matrix}(a_1,\alpha_1).........,(a_A,\alpha_A)\\(b_1,\beta_1).........,\ (b_B,\beta_B)\end{matrix}\right.\right],$$

$A+1 \le j \le A+m$

In view of (1.1.1) and (1.3.2) this can be expressed as

$$\frac{1}{2\pi i}\int_C \frac{\prod_{j=1}^{m}G\left(q^{b_j-\alpha_j s}\right)\prod_{j=1}^{n}G\left(q^{1-a_j+\beta_j s}\right)\left(1-q^{-b_j+\beta_j s}\right)\pi\left(\frac{u_{A+1}^{\beta_1}.......u_{A+B}^{\beta_B}}{u_1^{\alpha_1}......u_A^{\alpha_A}}.\frac{1}{u_{A+B+1}}\right)^s}{\prod_{j=m+1}^{B}G\left(q^{1-b_j+\alpha_j s}\right)\prod_{j=n+1}^{A}G\left(q^{a_j-\alpha_j s}\right)G\left(q^{1-s}\right)\sin\pi s}ds$$

$$u_1^{a_1-1}........u_A^{a_A-1}u_{A+1}^{-b_1}....u_j^{-(b_j+1)}u_{A+B}^{-b_B}$$

In view of (1.1.2), it takes the form

$$H_{A,B}^{m,n}\left[\frac{u_{A+1}^{\beta_1}.......u_{A+B}^{\beta_B}}{u_1^{\alpha_1}......u_A^{\alpha_A}}.\frac{1}{u_{A+B+1}};\right.$$

$$\left.q\left|\begin{matrix}(a_1,\alpha_1), \quad .... \quad (a_A,\alpha_A)\\(b_1,\beta_1)....,(b_l+1,\beta_j)......., (b_m,\beta_m),(b_{m+1},\beta_{m+1}),...,(b_B,\beta_B)\end{matrix}\right.\right]$$

$$u_1^{a_1-1}......u_A^{a_A-1}u_{A+1}^{-b_1}....u_{A+j}^{-(b_j+1)}u_{A+m}^{-b_m}u_{A+m+1}^{-b_{m+1}}....u_{A+B}^{-b_B}$$

With the help of (1.3.1), it becomes

$$\psi_{A,B}^{m,n}\left[\frac{u_{A+1}^{\beta_1}.......u_{A+B}^{\beta_B}}{u_1^{\alpha_1}......u_A^{\alpha_A}}.\frac{1}{u_{A+B+1}};\right.$$

$$\left. q\left|\begin{matrix}(a_1,\alpha_1), & .... & (a_A,\alpha_A)\\ (b_1,\beta_1)....,(b_j+1,\beta_j)......,(b_m,\beta_m),(b_{m+1},\beta_{m+1}),...,(b_B,\beta_B)\end{matrix}\right.\right]$$

$$; A+1\le j\le A+m$$

= R.H.S.

This completes the proof.

Proof of (iv) We consider, L.H.S.

$$\Delta_j^{+}\psi_{A,B}^{m,n}\left[\frac{u_{A+1}^{\beta_1}.......u_{A+B}^{\beta_B}}{u_1^{\alpha_1}......u_A^{\alpha_A}}.\frac{1}{u_{A+B+1}};q\left|\begin{matrix}(a_1,\alpha_1)........,(a_A,\alpha_A)\\ (b_1,\beta_1).........,\ (b_B,\beta_B)\end{matrix}\right.\right],$$

$$A+m+1\le j\le A+B$$

In view of (1.1.1) and (1.3.2) this can be expressed as

$$\frac{1}{2\pi i}\int_C\frac{\prod_{j=1}^{m}G\left(q^{b_j-\beta_j s}\right)\prod_{j=1}^{n}G\left(q^{1-a_j+\alpha_j s}\right)\left(1-q^{-b_j+\beta_j s}\right)\pi\left(\frac{u_{A+1}^{\beta_1}.......u_{A+B}^{\beta_B}}{u_1^{\alpha_1}......u_A^{\alpha_A}}.\frac{1}{u_{A+B+1}}\right)^s}{\prod_{j=m+1}^{B}G\left(q^{1-b_j+\beta_j s}\right)\prod_{j=n+1}^{A}G\left(q^{a_j-\alpha_j s}\right)G\left(q^{1-s}\right)\sin\pi s}ds$$

$$u_1^{a_1-1}........u_A^{a_A-1}u_{A+1}^{-b_1}....u_{A+m}^{-b_m}\,u_{A+m+1}^{-b_{m+1}}u_j^{-(b_j+1)}u_{A+B}^{-b_B}$$

In view of (1.1.2), it takes the form

$$H_{A,B}^{m,n}\left[\frac{u_{A+1}^{\beta_1}.......u_{A+B}^{\beta_B}}{u_1^{\alpha_1}......u_A^{\alpha_A}}.\frac{1}{u_{A+B+1}};\right.$$

$$\left. q\left|\begin{matrix}(a_1,\alpha_1), & .... & (a_A,\alpha_A)\\ (b_1,\beta_1).........,(b_m,\beta_m),(b_{m+1},\beta_{m+1})....,(b_l+1,\beta_j)...,(b_B,\beta_B)\end{matrix}\right.\right]$$

$$u_1^{a_1-1}......u_A^{a_A-1}u_{A+1}^{-b_1}u_{A+m+1}^{-b_{m+1}}....u_{A+j}^{-(b_j+1)}....u_{A+B}^{-b_B}$$

With the help of (1.3.1), it becomes

$$\psi_{A,B}^{m,n}\left[\frac{u_{A+1}^{\beta_1}.......u_{A+B}^{\beta_B}}{u_1^{\alpha_1}......u_A^{\alpha_A}}.\frac{1}{u_{A+B+1}};\right.$$

$$q\left|\begin{matrix}(a_1,\alpha_1), & \ldots\ldots\ldots\ldots\ldots & (a_A,\alpha_A)\\ (b_1,\beta_1)\ldots\ldots,(b_m,\beta_m),(b_{m+1},\beta_{m+1})\ldots,(b_j+1,\beta_j),\ldots,(b_B,\beta_B)\end{matrix}\right]$$

$; A+m+1 \le j \le A+B$

= R.H.S.

This completes the proof.

Proof of (v) We consider, L.H.S.

$$\Delta^{+}_{A+B+1}\psi^{m,n}_{A,B}\left[\frac{u_{A+1}^{\beta_1}\ldots\ldots u_{A+B}^{\beta_B}}{u_1^{\alpha_1}\ldots\ldots u_A^{\alpha_A}}\cdot\frac{1}{u_{A+B+1}};q\left|\begin{matrix}(a_1,\alpha_1)\ldots\ldots\ldots,(a_A,\alpha_A)\\(b_1,\beta_1)\ldots\ldots\ldots,\ (b_B,\beta_B)\end{matrix}\right.\right],$$

In view of (1.1.1) and (1.3.2) this can be expressed as

$$\frac{1}{2\pi i}\int_C\frac{\prod_{j=1}^{m}G\left(q^{b_j-\beta_j s}\right)\prod_{j=1}^{n}G\left(q^{1-a_j+\alpha_j s}\right)\left(1-q^{-s}\right)\pi\left(\frac{u_{A+1}^{\beta_1}\ldots\ldots u_{A+B}^{\beta_B}}{u_1^{\alpha_1}\ldots\ldots u_A^{\alpha_A}}\cdot\frac{1}{u_{A+B+1}}\right)^s}{\prod_{j=m+1}^{B}G\left(q^{1-b_j+\beta_j s}\right)\prod_{j=n+1}^{A}G\left(q^{a_j-\alpha_j s}\right)G\left(q^{1-s}\right)\sin\pi s}ds$$

$$u_1^{a_1-1}\ldots\ldots u_A^{a_A-1}u_{A+1}^{-b_1}\ldots\ldots u_{A+B}^{-b_B}\,u_{A+B+1}^{-1}$$

Employing (1.1.2), it takes the form

$$\frac{1}{2\pi i}\int_C\frac{\prod_{j=1}^{m}G\left(q^{b_j-\beta_j s}\right)\prod_{j=1}^{n}G\left(q^{1-a_j+\alpha_j s}\right)\pi\left(\frac{u_{A+1}^{\beta_1}\ldots\ldots u_{A+B}^{\beta_B}}{u_1^{\alpha_1}\ldots\ldots u_A^{\alpha_A}}\cdot\frac{1}{u_{A+B+1}}\right)^s}{\prod_{j=m+1}^{B}G\left(q^{1-b_j+\beta_j s}\right)\prod_{j=n+1}^{A}G\left(q^{a_j-\alpha_j s}\right)G\left(q^{1-s}\right)\sin\pi s}ds$$

$$u_1^{a_1-1}\ldots\ldots u_A^{a_A-1}u_{A+1}^{-b_1}\ldots\ldots u_{A+B}^{-b_B}\,u_{A+B+1}^{-1}$$

Replacing s+1 by t, it becomes

$$\frac{1}{2\pi i}\int_C\frac{\prod_{j=1}^{m}G(q^{(b_j-\beta_j s)-\beta_j t})\prod_{j=1}^{n}G(q^{1-a_j+\alpha_j s})\pi\left(\frac{u_{A+1}^{\beta_1}\ldots\ldots u_{A+B}^{\beta_B}}{u_1^{\alpha_1}\ldots\ldots u_A^{\alpha_A}}\cdot\frac{1}{u_{A+B+1}}\right)^s}{\prod_{j=m+1}^{B}G(q^{1-(b_j+\beta_j)+\beta_j t})\prod_{j=n+1}^{A}G(q^{(a_j-\alpha_j)-\alpha_j t})G(q^{1-t})\sin\pi t}ds$$

$$u_1^{(a_1+\alpha_1)-1}\ldots\ldots u_A^{(a_A-\alpha_A)-1}u_{A+1}^{-(b_1+\beta_1)}\ldots\ldots u_{A+B}^{-(b_B+\beta_B)}$$

It can be further put as

$$-H_{A,B}^{m,n}\left[\frac{u_{A+1}^{\beta_1}\ldots\ldots u_{A+B}^{\beta_B}}{u_1^{\alpha_1}\ldots\ldots u_A^{\alpha_A}}\cdot\frac{1}{u_{A+B+1}};q\left|\begin{matrix}(a_1+\alpha_1,\alpha_1), & \ldots\ldots & (a_A+\alpha_A,\alpha_A)\\(b_1+\beta_1,\beta_1), & \ldots\ldots & (b_B+\beta_B,\beta_B)\end{matrix}\right.\right]$$

$$u_1^{(a_1+\alpha_1)-1}\ldots\ldots u_A^{(a_A-\alpha_A)-1}u_{A+1}^{-(b_1+\beta_1)}\ldots\ldots u_{A+B}^{-(b_B+\beta_B)}$$

$; A+m+1\le j\le A+B$

With help of (1.3.1), it becomes

$$-\psi_{A,B}^{m,n}\left[\frac{u_{A+1}^{\beta_1}\ldots\ldots u_{A+B}^{\beta_B}}{u_1^{\alpha_1}\ldots\ldots u_A^{\alpha_A}}\cdot\frac{1}{u_{A+B+1}};q\left|\begin{matrix}(a_1+\alpha_1,\alpha_1), & \ldots\ldots & (a_A+\alpha_A,\alpha_A)\\(b_1+\beta_1,\beta_1), & \ldots\ldots & (b_B+\beta_B,\beta_B)\end{matrix}\right.\right]$$

= R.H.S.

This completes the proof.

On making use of (1.3.3) of (1.3.7), equation (1.2.6) becomes the canonical q-difference equation

$$\left(\Delta_1^{-\alpha_1}\ldots\ldots\Delta_n^{-\alpha_n}\Delta_{n+1}^{+\alpha_{n+1}}\ldots\Delta_A^{+\alpha_A}\Delta_{A+B+1}^{+}+\Delta_{A+1}^{-\beta_1}\ldots\Delta_{A+m}^{-\beta_1}\Delta_{A+m+1}^{+\beta_{m+1}}\ldots\Delta_{A+m}^{-\beta_B}\right)\psi_{A,B}^{m,n}=0$$

... (1.3.8)

The q-difference operators $\Delta_j^-(1\le j\le n)$, $\Delta_j^+(n+1\le j\le A)$, $\Delta_j^-(A+1\le j\le A+m)$, $\Delta_j^+(A+m+1\le j\le A+B)$ and Δ_{A+B+1}^+ are symmetry operators for the canonical equation (1.3.8).

The detailed account of the q-difference equations, canonical equations and symmetry operators are available in the monograph due to Agrawal, Kalnins and Miller [1], Kalnins and Miller [2] and Miller [5,6].

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