IJETST- Vol.||03||Issue||11||Pages 4777-4783||November||ISSN 2348-9480 2016

Open access Journal International Journal of Emerging Trends in Science and Technology
IC Value: 76.89 (Index Copernicus) Impact Factor: 2.838 DOI: https://dx.doi.org/10.18535/ijetst/v3i11.09

Relation Ship between Classes of Univalent Functions

Author
Dr M.Aparna
Sr. Asst. Prof of Mathematics

Abstract
In this paper we investigated the relationship between f and $g(z)=g_{\gamma}(z)=z(f(z)/z)^{\gamma}$ *, We find sufficient conditions on f for g to be in* $S^{*}(\alpha(p,q))$, $S_1^{*}(\alpha(p,q))$ *or K. The sufficient conditions found by Reade Silverman, and Todorov.*
Keywords: *Univalent Function, Starlike Function, convex Function, Analytic Function.*

1. INTRODUCTION

A function

$$(1.1)\qquad f(z)=z+\sum_{n=2}^{\infty}a_n z^n$$

is said to be in the family S if it is analytic and univalent in the unit disk $\Delta=\{|z|<1\}$. The subfamily of functions starlike of order $\alpha\,(p,q)$, denoted by $S^{*}(\alpha(p,q))$ consists of functions f for which $\mathrm{Re}(zf'/f)\geq\alpha(p,q),\quad 0\leq\alpha(p,q),\leq 1,$ for $z\in\Delta$. We further denoted by $S_1^{*}(\alpha\,(p,q))$, the subfamily of $S^{*}(\alpha(p,q))$ consisting of functions f for which $|(zf'/f)-1|\leq 1-\alpha(p,q)),\ 0\leq\alpha(p,q)\leq 1,$ for $z\in\Delta$. It is known as a sufficient condition, for f to be in $S_1^{*}(\alpha(p,q))$ is

$$(1.2)\qquad \sum_{n=2}^{\infty}(n-\alpha(p,q))|a_n|\leq 1-\alpha(p,q)$$

and this condition is necessary if f is of the form

$$(1.3)\qquad f(z)=z-\sum_{n=2}^{\infty}a_n z^n\ ,\quad a_n\geq 0.$$

The subfamily of $S^{*}(\alpha(p,q))$ consisting of functions of the form (1.3) and is denoted by $T^{*}(\alpha(p,q))$ $(\subset S^{*}{}_1(\alpha(p,q)))$. Finally, a function f of the form .1.1) is said to be in K, if the family of convex functions $\mathrm{Re}(1+zf''/f')\geq 0$ for $z\in\Delta$.

In this paper we investigated the relationship between f and

$$(1.4)\qquad g(z)=g_{\gamma}(z)=z(f(z)/z)^{\gamma}$$

where γ is real.

When f is in S or in one of the above subclasses. In Theorem 2, We find sufficient conditions on f for g to be in $S^{*}(\alpha(p,q))$, $S_1^{*}(\alpha(p,q))$ or K. The sufficient conditions found by Reade Silverman, and Todorov for functions of the form $z\Big/\left(1+\sum_{n=1}^{\infty}b_n z^n\right)$ to be in $S^{*}(\alpha(p,q))$ or K follows from γ=-1 in (1.4). In

Theorem 11 we find necessary and sufficient conditions for g to be in $S^*(\alpha(p,q))$, $S_1^*(\alpha(p,q))$ or $T^*(\alpha(p,q))$ in terms of the corresponding f. This leads to the coefficient bounds on g. In Theorem 18, we show that $f \in S$ implies $g \in S$ only for γ=0,1.

In this chapter we have assumed unless otherwise stated, that f is of the form (1.1) with corresponding to g is of the form (1.4), γ is real. Note that the trivial case γ=0, where g(z)=z for any f.

2. Theorem: The function g is in $S^*(\alpha(p,q))$ if

$$(2.1) \qquad \sum_{n=2}^{\infty} |\gamma|(n-1) + |\gamma(n-1) + 2(1-\alpha(p,q))| |a_n| \le 2(1-\alpha(p,q)).$$

Proof: Let $p(z) = zg'(z)/g(z) = 1 + \gamma((zf'(z)/f(z)) - 1)$ and

$q(z) = [1-(p(z)-\alpha(p,q))/(1-\alpha(p,q)]/[1+(p(z)-\alpha(p,q))/(1-\alpha(p,q))]$.

Then $\operatorname{Re} p(z) \ge \alpha(p,q)$ for $z \in \Delta$ if and only if $|q(z)| \le 1$. But

$$q(z) = \left| \frac{-\gamma((zf'/f)-1)}{2(1-\alpha(p,q)) + \gamma((zf'/f)-1)} \right|$$

$$= \left| \frac{-\gamma \sum_{n=2}^{\infty} (n-1) a_n z^{n-1}}{2(1-\alpha(p,q)) + \sum_{n=2}^{\infty} [\gamma(n-1) + 2(1-\alpha(p,q)) a_n] z^{n-1}} \right|$$

$$\le \left| \frac{\sum_{n=2}^{\infty} |\gamma|(n-1)|a_n|}{2(1-\alpha(p,q)) - \sum_{n=2}^{\infty} |\gamma|(n-1) + 2(1-\alpha(p,q))|a_n|} \right|$$

Now from (2.1) is equivalent to this last expression being bounded above by (1.1).

The proof is completed.

The conclusion of Theorem 2 split into three cases, which are in the following corollary

3. Corollary: The function g is in $S^*(\alpha(p,q))$ if

$$\text{(i)} \quad \sum_{n=2}^{\infty} [\gamma(n-1) + (1-\alpha(p,q))] |a_n| \le 1-\alpha(p,q), \qquad \gamma > 0,$$

$$\text{(ii)} \quad \sum_{n=2}^{\infty} [|\gamma|(n-1) - (1-\alpha(p,q))] |a_n| \le 1-\alpha(p,q), \gamma \le -2(1-\alpha(p,q)),$$

$$\text{(iii)} \quad \sum_{n=n_0+1}^{\infty} [|\gamma|(n-1) - (1-\alpha(p,q))] |a_n| \le 1-\alpha(p,q) \left(1 - \sum_{n=2}^{n_0} |a_n| \right),$$

$$-2(1-\alpha(p,q)) < \gamma < 0,$$

where n_0 is smallest integer for which $n_0 \ge 2(1-\alpha(p,q))/|\gamma|$. Equality holds when $f(z) = z + (1-\alpha(p,q))z^n / [|\gamma|(n-1) + (1-\alpha(pq))]$ for $n \ge 2$ in the first two cases and for $n \ge n_0 + 1$ in the third case.

By the above theorem we have the following corollary

4. **Corollary:** The function $f(z)=z\Big/\left(1+\sum_{n=2}^{\infty}b_n z^n\right)$ is in $S^*(\alpha(p,q))$, $0\le\alpha(p,q)\le 1$, if

$$\sum_{n=2}^{\infty}(n-1+\alpha(p,q))|b_n|\le \begin{cases}(1-\alpha(p,q))-(1-\alpha(p,q))|b_1|, & 0\le\alpha(p,q)\le\frac{1}{2},\\ (1-\alpha(p,q))-\alpha(p,q)|b_1|, & \frac{1}{2}\le\alpha(p,q)\le 1.\end{cases}$$

Proof: By Putting $\gamma=-1$ and $a_n=b_n$ in Corollary 3 we get the proof of this corollary.

By putting p=1, q=0 in the Theorem 2 we have the following corollary

5. Corollary: The function $f(z)=z\Big/\left(1+\sum_{n=2}^{\infty}b_n z^n\right)$ is in $S^*(\alpha)$, $0\le\alpha\le 1$, if

$$\sum_{n=2}^{\infty}(n-1+\alpha)|b_n|\le \begin{cases}(1-\alpha)-(1-\alpha)|b_1|, & 0\le\alpha\le\frac{1}{2},\\ (1-\alpha)-\alpha|b_1|, & \frac{1}{2}\le\alpha\le 1.\end{cases}$$

Proof: By Putting $\gamma=-1$, $a_n=b_n$, p=1, q=0 in Corollary 3 we get the proof.

6. Theorem: The function g is in $S_1^*(\alpha(p,q))$ if

(6.1) $$\sum_{n=2}^{\infty}[|\gamma|(n-1)+(1-\alpha(p,q))]|a_n|\le 1-\alpha(p,q),$$

which is equal to

$$f(z)=z+(1-\alpha(p,q))z^n/[|\gamma|(n-1)+(1-\alpha(pq))],\quad n\ge 2\ .$$

Proof: We have

$$\left|\frac{zg'}{g}-1\right|=\left|\gamma\left(\frac{zf'}{f}-1\right)\right|=\left|\frac{\sum_{n=2}^{\infty}\gamma(n-1)a_n z^{n-1}}{1+\sum_{n=2}^{\infty}a_n z^{n-1}}\right|$$

$$\le\left|\frac{|\gamma|\sum_{n=2}^{\infty}(n-1)|a_n|}{1-\sum_{n=2}^{\infty}|a_n|}\right|\le 1-\alpha(p,q)$$

if and only if (6.1) holds.

Remark: Since $S_1^*(\alpha(p,q))\subset S^*(\alpha(p,q))$. The special case $\gamma>0$ in Theorem 2 is a consequence of Theorem 6.

7. Theorem: The function g is in K if for some a, $0\le a\le 1$,

(i) $$\sum_{n=2}^{\infty}[|\gamma-1|(n-1)+a]|a_n|\le a$$ and

(ii) $$\sum_{n=2}^{\infty}(n-a)|\gamma(n-1)+1||a_n|\le 1-a.$$

Proof: Write

$$r(z)=1=zg''(z)/g'(z)$$

$$=1+\frac{(\gamma-1)\sum_{n=2}^{\infty}(n-1)a_n z^{n-1}}{1+\sum_{n=2}^{\infty}a_n z^{n-1}}+\frac{\sum_{n=2}^{\infty}(n-1)(\gamma(n-1)+1)a_n z^{n-1}}{1+\sum_{n=2}^{\infty}[\gamma(n-1)+1]a_n z^{n-1}}$$

A sufficient condition for $\operatorname{Re} r(z) \geq 0$ in Δ is

$$\left| \frac{\sum_{n=2}^{\infty} (\gamma-1)(n-1) a_n z^{n-1}}{1+\sum_{n=2}^{\infty} a_n z^{n-1}} \right| \leq \frac{\sum_{n=2}^{\infty} |\gamma-1|(n-1)|a_n|}{1-\sum_{n=2}^{\infty} |a_n|} \leq a$$

and the inequality

$$\left| \frac{\sum_{n=2}^{\infty} (n-1)(\gamma(n-1)+1) a_n z^{n-1}}{1+\sum_{n=2}^{\infty} (\gamma(n-1)+1) a_n z^{n-1}} \right| \leq \frac{\sum_{n=2}^{\infty} (n-1)|\gamma(n-1)+1||a_n|}{1-\sum_{n=2}^{\infty} |\gamma(n-1)+1||a_n|} \leq 1-a$$

holds for some a, $0 \leq a \leq 1$. But these inequalities are equivalent to (i) and (ii) being satisfied.
The proof is completed.
When $\gamma > 0$ we may choose the real number a in Theorem 7, so that

(7.1). $[(n-1)|\gamma-1| + a]/a \leq (n-a)(\gamma(n-1)+1)/(1-a)$

holds for $n \geq 2$,
which means that condition (ii) implies condition (i). Choosing the smallest value of a, it leads to the following corollary which states
8. Corollary: The function g is in K if

$$\sum_{n=2}^{\infty} [\gamma n^2 + (1-\gamma-\gamma a)n - a(1-\gamma)]|a_n| \leq 1-a,$$

where $a = \left(2+\gamma-\sqrt{(5\gamma^2+4)}\right)/2\gamma$ when $0 < \gamma \leq 1$ and $a = 3/2 - \sqrt{5/4 + 1/\gamma}$ when $\gamma > 1$. Equality holds for $f(z) = z + (1-a)z^2/(\gamma+1)(2-a)$.

Remark: The special case $\gamma = 1$ in Theorem 2 and

Theorem 6 reduces to the sufficient condition (1.2) for f to be in $S^*(\alpha(p,q))$ and $S_1^*(\alpha(p,q))$ respectively, and to the well known sufficient condition for convexity, $\sum_{n=2}^{\infty} n^2 |a_n| \leq 1$, in Corollary 3 of the

Theorem 7.
When $\gamma < 0$, in (7.1) need not hold for all n. In fact the right side of (1.3) vanishes when $\gamma = -1/(n-1)$. For $\gamma < -\frac{1}{2}$ we can still find a expression in which, for some a, inequality (7.1) holds for $n \geq 3$ but not for n=2.
This leads to the following Corollary which states
9. Corollary: The function g is in K for $\gamma < -\frac{1}{2}$ if

$$((1+|\gamma|+a)/a)|a_2| + \sum_{n=3}^{\infty} (n-a)[|\gamma|(n-1)-1]/(1-a) \leq 1,$$

$$\text{where} \qquad a = \begin{cases} \left(\sqrt{12\gamma^2+8\gamma+5}+1+4\gamma\right)/2(1+\gamma), & \gamma \neq -1, \\ 2/3 & \gamma = -1. \end{cases}$$

By above theorem we have the following corollary

10. Corollary: The function $f(z) = z/(1+\sum_{n=1}^{\infty} b_n z^n)$ is in K if

$$4|b_1| + \sum_{n=2}^{\infty} (n-1)(3n+1)|b_n| \le 1.$$

Proof: By Putting $\gamma = -1$, $a = \frac{2}{3}$ and $a_n = b_{n-1}$, p=1, q=0 in Corollary 9 we get the proof.

11. Theorem:

(i) The function f is in $S^*(\alpha(p,q))$ if and only if $g \in S^*(1-\gamma(1-\alpha(p,q)))$, $0 \le \gamma(1-\alpha(p,q)) \le 1$, and $g \in S^*(\alpha(p,q))$ if and only if $f \in S^*(1-(1-\alpha(p,q))/\gamma)$, $(1-\alpha(p,q))/\gamma \le 1$;

(ii) The function f is in $S_1^*(\alpha(p,q))$ if and only if $g \in S_1^*(1-|\gamma|(1-\alpha(p,q)))$, $|\gamma|(1-\alpha(p,q)) \le 1$, and $g \in S_1^*(\alpha(p,q))$

if and only if $f \in S_1^*(1-(1-\alpha(p,q))/|\gamma|)$, $(1-\alpha(p,q))/|\gamma| \le 1$.

Proof: The first result follows from the identity $zg'/g = 1 + \gamma((zf'/f)-1)$ and the second follows from the identity $|(zg'/g)-1| = |\gamma||(zf'/f)-1|$.

By the above theorem we have the following corollary

12. Corollary: For $0 \le \gamma \le 1$, $g \in S^*(\alpha(p,q))$ whenever $f \in S^*(\alpha(p,q))$ and for $|\gamma| \le 1$, $g \in S_1^*(\alpha(p,q))$ whenever $f \in S_1^*(\alpha(p,q))$.

The coefficient bounds on f in $S^*(\alpha(p,q))$ and $S_1^*(\alpha(p,q))$ lead to corresponding coefficient bounds on g. Also by the above theorem we have the following corollary

13. Corollary: If $f \in S^*(\alpha(p,q))$, $0 \le \gamma(1-\alpha(p,q)) \le 1$, and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$, then

$|b_n| = \prod_{k=2}^{n}[(k-2) + 2\gamma(1-\alpha(p,q))]/(n-1)!$. Equality holds for $g_0(z) = z(f_0(z)/z)^{\gamma}$, where $f_0(z) = z/(1-z)^{2(1-\alpha(p,q))}$.

Proof: The function $f_0(z)$ is to maximize the coefficients of functions in $S^*(\alpha(p,q))$, $0 \le \alpha(p,q) \le 1$.

By Putting p=1, q=0 in Theorem 11 we have the following Corollary

14. Corollary: If $f \in S^*(\alpha)$, $0 \le \gamma(1-\alpha) \le 1$, and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$, then

$|b_n| = \prod_{k=2}^{n}[(k-2) + 2\gamma(1-\alpha)]/(n-1)!$. Equality holds for $g_0(z) = z(f_0(z)/z)^{\gamma}$, where $f_0(z) = z/(1-z)^{2(1-\alpha)}$.

Proof: The function $f_0(z)$ is to maximize the coefficients of functions in $S^*(\alpha)$, $0 \le \alpha \le 1$.

By the above theorem we have the following corollary

15. Corollary: If $f \in S_1^*(\alpha(p,q))$, $|\gamma|(1-\alpha(p,q)) \le 1$, and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$, then

$|b_n| \le |\gamma|(1-\alpha(p,q))/(n-1)$.. Equality holds for $g_n(z) = z(f_n(z)/z)^{\gamma}$, where $f_n(z) = z\exp((1-\alpha(p,q))z^{n-1}/(n-1))$.

Proof: In the function $f_n(z)$ was shown to maximize the nth coefficient for functions in $S_1^*(\alpha(p,q))$, $0 \le \alpha(p,q) \le 1$.

In the previous corollaries the extremal f in $S^*(\alpha(p,q))$ and $S_1^*(\alpha(p,q))$ was transformed into g that was extremal in $S^*(1-\gamma(1-\alpha(p,q)))$ and $S_1^*(1-|\gamma|(1-\alpha(p,q)))$, respectively. This made the determination of coefficient bounds on g straight forward. We now consider a special subclass of $S_1^*(\alpha(p,q))$ for which this is not the case. Since $T^*(\alpha(p,q)) \subset S^*{}_1(\alpha(p,q))$, it follows from Theorem 8 if $f \in T^*(\alpha(p,q))$ then $g \in S_1^*(1-|\gamma|(1-\alpha(p,q)))$, $|\gamma|(1-\alpha(p,q)) \le 1$. The extremal functions g_n for the coefficients, however, were not associated with corresponding functions $f \in T^*(\alpha(p,q))$. To determine such coefficient bounds. we need the Lemma 16 .

16. Lemma: If $\left(1+\sum_{n=2}^{\infty} a_n z^n\right)^{\gamma} = 1+\sum_{n=2}^{\infty} b_n z^n$ is analytic in a neighborhood of the origin γ real, then

$$b_{k+1} = \sum_{i=0}^{k} [\gamma-(\gamma+1)j/(k+1)]a_{k+1-j}b_j \quad (k=0,1,\ldots\ldots\ldots,; b_0=1).$$

17. Theorem: If $f(z) = z - \sum_{n=2}^{\infty} a_n z^n \in T^*(\alpha(p,q))$, $|\gamma|(1-\alpha(p,q)) \le 1$, and

$g(z) = z(f(z)/z)^{\gamma} = z + \sum_{n=2}^{\infty} b_n z^n$, then $|b_n| \le |\gamma|(1-\alpha(p,q))/(n-\alpha(p,q))$.

Equality holds for $g_n(z) = z(f_n(z)/z)^{\gamma}$, where $f_n(z) = z-(1-\alpha(p,q))z^n/(n-\alpha(p,q))$.

Proof: By the Lemma 7.16, $b_2 = \gamma a_2$ and

(17.1). $$b_{k+1} = \gamma a_{k+1} + \sum_{j=1}^{k-1} [\gamma-(\gamma-1)j/k]a_{k+1-j}b_{j+1.}$$

From (1.2) we may get $a_k = \lambda_k(1-\alpha(p,q))/(k-\alpha(p,q))$ with $\sum_{k=2}^{\infty}\lambda_k \le 1$ and write (17.1) as

$b_{k+1} = \gamma\lambda_k((1-\alpha(p,q)/(k+1-\alpha(p,q)) + \sum_{j=1}^{k-1}[\gamma-((\gamma+1)j/k)]$

$\lambda_{k+1-j}((1-\alpha(p,q))/(k+1-j-\alpha(p,q)))b_{j+1}$. It suffices to show that $|b_{k+1}|$ is uniquely maximized when $\lambda_{k+1}=1$, which is true if

(17.2) $$|\gamma|\left(\frac{1-\alpha(p,q)}{k+1-\alpha(p,q)}\right) > \left|\gamma-\frac{(\gamma+1)j}{k}\right|\left(\frac{1-\alpha(p,q)}{k+1-j-\alpha(p,q)}\right)|b_{j+1}|,$$
$$1 \le j \le k-1.$$

Since $|b_2| = \gamma\lambda_2(1-\alpha(p,q))/(2-\alpha(p,q)) \le |\lambda|(1-\alpha(p,q))/(2-\alpha(p,q))$,

We may assume that

(17.3) $|b_j| \le |\lambda|(1-\alpha(p,q))/(j-\alpha(p,q))$ for j=1,2,………,k.

Note that

(17.4) $$\left|\gamma-\frac{(\gamma+1)j}{k}\right| \le \frac{|\gamma|(k-j)+j}{k} \le \frac{(k-j)+j(1-\alpha(p,q))}{k(1-\alpha(p,q))}$$
$$\le \frac{k-\alpha(p,q)}{k(1-\alpha(p,q))}$$

Substituting the upper bounds of (17.3) and (17.4) into the right side of (17.2) we get

(17.5) $$|\gamma|\left(\frac{1-\alpha(p,q)}{k+1-\alpha(p,q)}\right)$$

$$> \left(\frac{k-\alpha(p,q)}{k(1-\alpha(p,q))}\right)\left(\frac{1-\alpha(p,q)}{k+1-j-\alpha(p,q)}\right)\left(\frac{|\gamma|(1-\alpha(p,q))}{j+1-\alpha(p,q)}\right)$$

Since the right side of (17.5) is maximized when j=1, inequality (17.5) will be true if $1/(k+1-\alpha(p,q)) > 1/k(2-\alpha(p,q))$, which is valid for k>1.

This completes the proof of the theorem.

Remark: If $0 \le \gamma \le 1$, then $g(z) = z\left(1-\sum_{n=2}^{\infty} a_n z^{n-1}\right)^{\gamma} = z\left[1-\gamma\left(\sum_{n=2}^{\infty} a_n z^{n-1}\right)-(\gamma(1-\gamma)/2!)\left(\sum_{n=2}^{\infty} a_n z^{n-1}\right)^2 ---\right]$

$$= z - \sum_{n=2}^{\infty} b_n z^n, \qquad b_n \ge 0.$$

Thus, in addition to g being in $S_1^*(1-\gamma(1-\alpha(p,q)))$, we have $g \in T^*(1-\gamma(1-\alpha(p,q)))$.

While $g_\gamma(z) = z(f(z)/z)^{\gamma}$ usually it seems to share many of the nice properties of f, at least when f is in different subclasses of S, the same does not hold when the only restriction on f is that it is a member of S. In this case, g_γ need not be locally univalent.

18. Theorem: For every γ real, $\gamma \ne 0,1$. there exists an $f \in S$ for which $g_\gamma(z) = z(f(z)/z)^{\gamma} \notin S$.

Proof: We have

$$g'(z) = \left(\frac{f(z)}{z}\right)^{\gamma-1}\left[\frac{(1-\gamma)f(z)+\gamma z f'z}{z}\right] = 0$$

if $zf'(z)/f(z) = (\gamma-1)/\gamma$. Since zf'/f maps Δ onto the right half plane when $f(z) = z/(1-z)^2$, the corresponding g_γ will not be univalent when $\gamma < 0$ or $\gamma > 1$. Now we consider $\gamma \in (0,1)$. For every fixed $z \in \Delta$, the region of values of $\log(zf'(z)/f(z))$ for $f \in S$ is the disk $|w| \le \log((1+|z|)/(1-|z|))$. In particular, for any real number t we can find $z \in \Delta$ and $f \in S$ for which $\log(zf'(z)/f(z)) = t + \pi i$. Thus, $zf'(z)/f(z)) = -e^t = (\gamma-1)/\gamma$ when $t = \log((\gamma-1)/\gamma)$. This completes the proof of the theorem.

REFERENCES

1. Duren P.L.: Univalent Functions, Springer-Verlag Berlin Heidelberg and Tokyo,1983.
2. J.S.Jack: "Functions starlike and convex of order α". J.London Math Soc. 3,(1971), 469 –474.
3. Conway J.B, Functions of complex variables, Narosa Publishing House, 1985.
4. D.R.Wilken and J.Feng, "A remark on convex and starlike functions", Proc.Amer.Math. Soc, (1979), 287-290.
5. V.Ravichandran, C.Selvaraj and R.Rajalaksmi, "Sufficient conditions for starlike functions of order α", and Appl.Math, 1443-5756.
6. Herb Silverman, "Convex and starlike criteria", Internat.J.Math. & Math. Sci,Vol. 22, No.1(1999) ,75-79.
7. H.Silverman, "Univalent functions with negative coefficients", Proc. Amer. Math. Soc. 51(1975), 109-116.