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Signed Total Domatic Number of Directed Circulant Graphs

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Abstract
A function f : V(D) → {−1, 1} is a signed total dominating function (STDF) of a directed graph D, if forevery vertex v ∈ V, $f\left(N^-(v)\right) = \sum_{u \in N^-(v)} f(u) \geq 1$. A STDF of a directed graph D is said to be SETDF iffor every vertex v ∈ V, $f\left(N^-(v)\right)$ = 1 when $|N^-(v)|$ is odd and $f\left(N^-(v)\right)$ = 2 when $|N^-(v)|$ is even. Westudy some properties of signed total domatic number d_{st}(D) in directed circulant graphs. We characterizesome classes of directed circulant graphs for which $d_{st}(D) = \delta^-(D)$. Further, we find a necessary andsufficient condition for the existence of SETDF in a family of directed circulant graphs in terms of coveringprojection
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Introduction

Consult [2] and [3] for notation and terminology which are not defined here. Let D be a finite and simple digraph with vertex set V(D) = V and arc set E(D) = E. For every vertex v∈V(D), the in-set of v and the out set of v are defined by $N^-(v) = N_D^-(v)$ = {u ∈ V : (u, v) ∈ E} and $N^+(v) = N_D^+(v)$ = {u ∈ V : (v, u) ∈ E} respectively. For a vertex v ∈ V, $d_D^+(v) = d^+(v) = |N^+(v)|$ and $d_D^-(v) = d^-(v) = |N^-(v)|$ respectively denote the outdegree and indegree of the vertex v. The minimum and maximum indegree of D are denoted by $\delta^-(D)$ and $\Delta^-(D)$ respectively. Similarly the minimum and maximum outdegree of D are denoted by $\delta^+(D)$ and $\Delta^+(D)$ respectively.

The concept of signed domination number in undirected graphs has been introduced by J.E. Dunbar et al [3]. The concept of signed total domination number in undiretedd graphs has been introduced by Bohdan Zeinka[2]. In 2005, Bohdan Zelinka [1] study the concept of signed domination for directed graphs.

A function f : V → {−1, 1} is a signed dominating function (SDF) of a directed graph D, if for every vertex v ∈ V, $f(N^-[v]) = \sum_{u \in N^-[v]} f(u) \geq 1$ [4]. The signed domination number, denoted by γ_S(D), is the minimum weight of a signed dominating function of D [4]. A function f : V → {−1, 1} is a signed total dominating function (STDF) of a directed graph D, if for every vertex v ∈ V, $f\left(N^-(v)\right) = \sum_{u \in N^-(v)} f(u) \geq$ 1. The weight of the function f is defined as $w(f) = U_{v \in V} f(v)$. The signed total domination number, denoted γ_{st}(G), of D is the minimum weight of a signed total dominating function on D.

In this paper, we introduce the concept of signed efficient total dominating function (SETDF) for directed graphs. A STDF of a directed graph D is said to be SETDF if for every vertex v ∈ V, $f\left(N^-(v)\right)$ = 1 when $|N^-(v)|$ is odd and $f\left(N^-(v)\right)$ = 2 when $|N^-(v)|$ is even. A set {f_1, f_2, . . ., f_d} of signed dominating functions on a graph (directed graph) G with the property that $\sum_{i=1}^{d} f_i(x) \leq 1$ for each vertex x ∈ V(G), is called a signed dominating family on G. The maximum number of functions in a signed dominating family on G is the signed domatic number of G, denoted by $d_s(G)$.

A set $\{f_1, f_2, \ldots, f_d\}$ of signed total dominating functions on a graph (directed graph) G with the property that $\sum_{i=1}^{d} f_i(x) \leq 1$ for each vertex $x \in V(G)$, is called a signed total dominating family on G. The maximum number of functions in a signed total dominating family on G is the signed total domatic number of G, denoted by $d_{st}(G)$.

The signed domatic number of undirected and simple graphs was introduced by Volkmann and Zelinka [7]. They determined the signed domatic number of complete graphs and complete bipartite graphs. Further, they obtained some bounds for domatic number.

In this paper, we study the signed total domination number and signed total domatic number of directed circulant graphs *Cir(n, A)* for some generating set *A*. Further, we obtain a necessary and sufficient condition for the existence of SETDF in *Cir(n, A)* in terms of covering projection.

The following results can be found in [6].

Theorem 1.1 *[6] Let D be a directed graph of order n with signed total domination number $\gamma_{st}(D)$ and signed total domatic number $d_{st}(D)$.*
Then $\gamma_{st}(D).\ d_{st}(D) \leq n$.

Theorem 1.2 *[6] Let D be a directed graph with minimum in degree $\delta^-(D) \geq 1$, then $1 \leq d_{st}(D) \leq \delta^-(D)$.*

Theorem 1.3 *Let D be a directed graph such that $d^+(x) = d^-(x) = 2g + 1$ for each $x \in V$ and let $u \in V(D)$. If $d = d_{st}(D) = 2g + 1$ and $\{f_1, f_2, \ldots, f_d\}$*
Is a signed total domatic family of D, then $\sum_{i=1}^{d} f_i(u) = 1$ and $\sum_{u \in N^-(v)} f_i(x) = 1$ for each $u \in V(D)$ and each $1 \leq i \leq 2g + 1$.

Proof: Since $\sum_{i=1}^{d} f_i(u) \leq 1$, this sum has at least g summands which have the value -1. Since $\sum_{x \in N^-(v)} f_i(x) \geq 1$ for each $1 \leq i \leq 2g + 1$, this sum has atleast g + 1 summands which have the value 1. Also the sum $\sum_{x \in N^-(v)} \sum_{i=1}^{d} f_i(x) = \sum_{i=1}^{d} \sum_{x \in N^-(v)} f_i(x)$ has at least d.g summands of value -1 and at least d.(g+1) summands of value 1. Since the sum $\sum_{x \in N^-(v)} \sum_{i=1}^{d} f_i(x) = \sum_{i=1}^{d} \sum_{x \in N^-(v)} f_i(x)$ contains exactly d(2g+1) summands, it is easy to observe that $\sum_{i=1}^{d} f_i(u)$ have exactly g summands of value -1 and $\sum_{x \in N^-(v)} f_i(x)$ has exactly g+1 summands of value 1 for each $1 \leq i \leq d$. Hence we must have $\sum_{i=1}^{d} f_i(u) = 1$ and $\sum_{x \in N^-(v)} f_i(x) = 1$ for each $u \in V(D)$ and for each $1 \leq i \leq 2g + 1$.

2 γ_{st}(D) and SETDF in directed circulant graphs

Let Γ be a finite group and e be the identity element of Γ. A generating set of Γ is a subset *A* such that every element of *A* can be written as a product of finitely many elements of *A*. Assume that $e \notin A$ and $a \in A$ Implies $a^{-1} \in A$. Then the corresponding *Cayley graph* is a graph $G = (V, E)$, where $V(G) = \Gamma$ and $E(G) = \{(x, y)_a | x,y \in V(G), y = xa \text{ for some } a \in A\}$, denoted by *Cay*(Γ,*A*). It may be noted that G is connected regular graph degree of degree |*A*|. A Cayley graph constructed out of a finite cyclic group $(Z_n, \oplus_n)$ is called a circulant graph and it is denoted by *Cir(n, A)*, where A is a generating set of Z_n. When we leave the condition that $a \in A$ implies $a^{-1} \in A$, then we get directed circulant graphs. In a directed circulant graph *Cir(n, A)*, for every vertex v, $|N^-[v]| = |N^+[v]| = |A|+1$.

Throughout this section, $n(\geq 3)$ is a positive integer, $\Gamma = (Z_n, \oplus_n)$, where $Z_n = \{0, 1, 2, \ldots, n-1\}$ and D = *Cir(n, A)*, where $A = \{1, 2, \ldots, r\}$ and $1 \leq r \leq n-1$. From here, the operation $\oplus_n$ stands for modulo n addition in Z_n. In this section, we characterize the circulant graphs for which $d_{st}(D) = \delta^-(D)$. Also we find a necessary and sufficient condition for the existence of SEDF in *Cir(n, A)* in terms of covering projection.

Theorem 2.1 *Let $n \geq 3$ and $1 \leq r \leq n - 1$ (r is odd) be integers and $D = Cir(n, \{1, 2, . . ., r\})$ be a directed circulant graph. Then $d_{st}(D) = r$ if, and only if, r divides n.*

Proof: Assume that $d_{st}(D) = r$ and $\{f_1, f_2, . . ., f_r\}$ is a signed total domatic family on D. Note that $d^+(v) = d^-(v) = r$, for all $v \in V(D)$ and for each $v \in V(G)$, N(v) is a set of r(odd) consecutive integers. Thus $\gamma_{st}(D) \geq \frac{n}{r}$. Since $d_{st}(D) = r$, by Theorem 1.1, we have $\gamma_{st}(D) \geq \frac{n}{r}$ .Hence $\gamma_{st}(D) \geq \frac{n}{r}$. Suppose n is not a multiple of r. Then $n = kr + i$ for some $1 \leq i \leq r-1$.Let
$t = \gcd(i, r)$. Then there exist relatively prime integers p and q such that
$r = qt$ and $i = pt$.
Let a and b be the smallest positive integers such that $ar = bn$. Then $\gcd(a, b) = 1$(otherwise a and b will not be the smallest) and so the subgroup $< r >$ of the finite cyclic group Z_n, generated by r, must have a elements. Now $aqt = ar = b(kr + i) = b(kqt + pt)$ ~~= bt(~~kq + p). That is
$aq = b(kp + q)$. Since $\gcd(a, b) = \gcd(p, q) = 1$, $a = kp + q$ and $b = q$. Thus the subgroup $<r>$ must have $kp + q$ elements. But $t = \frac{r}{q} = \frac{n}{kp+q}$. Thus the subgroup $<t>$ of Z_n, generated by the element t, also have $kp + q$ elements and hence $<t> = <r>$. Since $d_s(D) = r$ and $\{f_1, f_2, . . ., f_r\}$ is a signed total domatic family of D, by Theorem 1.3, we have $\sum_{i=1}^{d} f_i(u) = 1$
and $\sum_{x \in N^-(v)} f_i(x) = 1$ and each $1 \leq i \leq r$. From the above fact and since $|N^-(v)| = r$ and $(f(N^-(v)) = 1$ (since r is odd) for all $v \in V(D)$, it is follows that if $f(a) = +1$, then $f(a \oplus_n r) = +1$ and if $f(a) = -1$, then $f(a \oplus_n r) = -1$. Thus the labelings of all the elements of the subgroup $<t>$ and the labelings of all the elements in each of the co-set of $<t>$ are same. By Lagranges theorem on subgroups, Z_n can be written as the union of co-sets of $<t> = <r>$. This means that $\gamma_{st}(D)$ must be a multiple of the number of elements of $<t>$, that is a multiple of (n/t) (since n is a multiple of t). Since $t < r$, it follows that $\frac{n}{r} < \frac{n}{r} \leq \gamma_{st}(D)$, a contradiction to $\gamma_{st}(D) = \frac{n}{r}$.
Conversely suppose r divides n. By theorem 1.2, $d_{st}(D) \leq r$. Let $r = 2g + 1$ for some integer $g \geq 1$. Define a STDF f_1 by $f_1(ir + 1) = f_1(ir + 2) =. . . = f_1(ir+(g+1)) = +1$ and $f_1(ir+(g+2)) = f_1(ir+(g+3)) = . . . = f_1(ir+(2g+1)) = -1$ for all $0 \leq i \leq \frac{n}{r} - 1$. Define $f_2(v) = f_1(v \oplus_n 1)$, $f_3(v) = f_1(v \oplus_n 2) = . . . = f_r(v) = f_1(v \oplus_n (r - 1))$. Then $\{f_1, f_2, . . . , f_r\}$ are STDFs on D with the property that $\sum_{i=1}^{d} f_i(x) \leq 1$ for each vertex $x \in V(D)$. Hence $d_s(D) \geq r$.

Example 2.2 Let n = 6 and r = 3. Then n is a multiple of r. Take the vertex set of D = *Cir*(6, {1, 2, 3}) as V(D) = {0, 1, 2, 3, 4, 5}. Also for r = 3, STDFs f_1, f_2 and f_3 (as discussed in the above theorem) of D are given below. $f_1(1) = +1$, $f_1(2) = +1$, $f_1(3) = -1$, $f_1(4) = +1$, $f_1(5) = +1$, $f_1(0) = -1$;
$f_2(1) = +1$, $f_2(2) = -1$, $f_2(3) = +1$, $f_2(4) = +1$, $f_2(5) = -1$, $f_2(0) = +1$;
$f_3(1) = -1$, $f_3(2) = +1$, $f_3(3) = +1$, $f_3(4) = -1$, $f_3(5) = +1$, $f_3(0) = +1$.

Theorem 2.3 *Let $n \geq 3$ be an integer and $1 \leq r \leq n - 1$ be an odd integer. Let $D = Cir(n, \{1, 2, . . ., r\})$ be a directed circulant graph. If n is a multiple of r, then $\gamma_{st}(D) = \frac{n}{r}$.*

Proof: Assume that n is a multiple of r. As discussed in Theorem 2.1,
$\gamma_{st}(D) = \frac{n}{r}$. It remains to show that there exists a STDF f with the property that $f(D) = \frac{n}{r}$. Define a function $f : V(D) \rightarrow \{+1, -1\}$ such that
$f(ir + 1) = f(ir + 2) = . . . = f(ir + (g + 1)) = +1$ and $f(ir + (g + 2)) =$

f(ir + (g + 3)) = . . . = f(ir + (2g + 1)) = −1 for all $0 \leq i \leq \frac{n}{r} - 1$, where r=2g+1. It is clear that f is a STDF and f(D) = (g + 1) ($\frac{n}{r}$) - (g)($\frac{n}{r}$) = $\frac{n}{r}$.

A graph $\tilde{G}$ is called a covering graph of G with covering projection f : $\tilde{G} \rightarrow G$ if there is a surjection f : V($\tilde{G}$) → V (G) such that f $|_{N(\tilde{v})}$: N($\widetilde{v}$) → N(v) is a bijection for any vertex v ∈ V(G) with $\tilde{v} \in f^{-1}(v)$ [5].

In 2001, J.Lee has studied the domination parameters through covering projections [5]. In this paper, we introduce the concept of covering projection for directed graphs and we study the STDF through covering projections

A directed graph D is called a covering graph of another directed graph H with covering projection f : D → H if there is a surjection f : V(D) → V(H) such that $f|_{N^+(u)} : N^+(u) \rightarrow N^+(v)$ and $f|_{N^-(u)} : N^-(u) \rightarrow N^-(v)$ are bijections for any vertex v ∈ V(H) with $u \in f^{-1}(v)$.

Lemma 2.4 *Let f : D → H be a covering projection from a directed graph D on to another directed graph H. If H has a SETDF, then so is D.*

Proof: Let f : D → H be a covering projection from a directed graph D on to another directed graph H. Assume that H has a SETDF h : V(H) →
{1, −1}. Define a function g : V(D) → {1, −1} defined by g(u) = h(f(u)) for all u ∈ V (D). Since h is a function form V(H) to {1, −1} and f : V(D) → V(H), g is well defined. We prove that for the graph D, g is a SETDF.

Let u ∈ V(D) and assume that $|N^-(u)|$ is odd. Since f is a covering projection, $|N^-(u)|$ and $|N^-(f(u))|$ are equal. Also $f|_{N^-(u)} : N^-(u) \rightarrow N^-(f(u))$ is a bijection. Also for each vertex x ∈ $N^-(u)$, we have g(x) = h(f(x)). Since h($N^-(f(u))$) = 1, we have g($N^-(u)$) = 1. Similarly, we can prove that g($N^-(u)$) = 2 when u ∈ V(G) and $|N^-(u)|$ is even. Hence g is a SETDF on D.

Theorem 2.5 *Let D = Cir(n, {1, 2, . . . , r}), r = 2g + 1 for some integer g(≥1) and γ_s(D) = $\frac{n}{r}$ Then D has a SETDF if and only if, there exists a covering projection from D onto the graph H = Cir(r + 1, {1, 2, . . . , r}).*

Proof: Suppose D has a SETDF, say f. Then$\sum_{x \in N^-(v)} f(x) = 2$ for all u∈v(D). Thus we can have f(a $\oplus_n$ r) = ±1 when ever f(a) = ±1 respectively. Thus the elements of the subgroup <r>, generated by r have the same sign.

Suppose n is not a multiple of r, then n = i.r + j for some 1 ≤ j ≤ r − 1. Let t = gcd(r, j). As in the proof of Theorem 2.1, we have γ_{st}(D) > $\frac{n}{r}$, a contradiction. Hence n must be a multiple of r.

In this case, define F : D → H = Cir(r +1, {1, 2, . . . , r}), defined by F (x) = x (mod (r + 1)). Note that, $|N^-(x)| = |N^+(x)| = |N^-(y)| = |N^+(y)|$ = r for all x ∈ V(D) and y ∈ V(H). We prove that F is a covering projection.

Let x ∈ V(D). Then F(x) = x(mod (r + 1)) = i for some i ∈ V(H). Note that by the definition of D and H, $N^+(x)$ = {x $\oplus_n$ 1, x $\oplus_n$ 2, . . . , x $\oplus_n$ r} and
$N^+(i)$ = {i $\oplus_{r+1}$ 1, i $\oplus_{r+1}$ 2, . . ., i $\oplus_{r+1}$ r}. Also, for each 1 ≤ j ≤ r, we have F (x $\oplus_n$ j) = i $\oplus_{r+1}$ j.
Thus $F|_{N^+(x)} : N^+(x) \rightarrow N^+(F(x))$ is a bijection. Similarly, we can prove that $F|_{N^-(x)}: N^-(x) \rightarrow N^-(F(x))$ is also a bijection and hence F is a covering projection from D onto H.

Conversely, suppose there exists a covering projection F from D onto the graph H = *Cir*(r + 1, {1, 2, . . ., r}). Define h : V(H) → {+1, −1} defined by h(x) = −1 when 1 ≤ x ≤ g and h(x) = +1 when g + 1 ≤ x ≤ 2g + 1. Then h is a SETDF of H and hence by Lemma 2.4, G has a SETDF.

Theorem 2.6 *Let D = Cir(n, {1, 2, . . ., r}), r = 2g be an even integer which divides n. Then*
(a) If D admits SETDF, then γ_{st}(D) = $\frac{2n}{r}$.

(b) D has a SETDF if and only if, there exists a covering projection from D onto the graph H = Cir(r + 1, {1, 2, . . ., r}).

Proof: (a). Let f be a SETDF and v ∈ V(D). Since f($N^-(v)$) = 2 for all v∈V(D) and $N^-(v)$ = {s $\oplus_n$ 1, s $\oplus_n$ 2, . . ., s $\oplus_n$ r} is a set of r consecutive vertices, we must have f({s $\oplus_n$ 1, s $\oplus_n$ 2, . . ., s $\oplus_n$ r}) = 2.
Hence f(D) ≥ 2 $\frac{n}{r}$. It remains to show that there exists a SETDF f with the property that f(D) = 2 $\frac{n}{r}$.
Define f : V(D) → {+1, −1} such that f(ir + 1) = f(ir + 2) =. . .= f(ir + (g + 1)) = +1 and f(ir + (g + 2)) = f(ir + (g + 3)) = . . .= f(ir +(2g)) = −1 for 0 ≤ I ≤ $\frac{n}{r} - 1$. It is clear that f is a SETDF and f(D) = (g + 1) $\frac{n}{r}$ − (g - 1) $\frac{n}{r} = 2\ \frac{n}{r}$.
(b) Suppose D has a SETDF f. Define h : H → {+1,−1}, defined by
h(v) = f(v) for all v ∈ V(H). Since f($N^-(v)$) = f(V(H)) = 2 for all v ∈ V(H), h is a SETDF on H. Note that, $|N^-(x)|$ = $|N^+(x)|$ = r = 2g, an even integer, for all x ∈ (D). Thus $\sum_{x\in N^-[u]} f(x) = 2$ for all u∈ v(D) and hence f(a $\oplus_{r+1}$ r) = ±1 when ever f(a) = ±1 respectively. Thus the function F : D → H, defined by F (x) = x (mod (r + 1)) is a covering projection from D onto H.
Conversely suppose there exists a covering projection F from D onto the graph H. Define h : V (H) → {+1, −1} defined by h(x) = −1 when 0 ≤ x ≤ g and h(x) = +1 when g + 1 ≤ x ≤ 2g − 1. Note that $\sum_{x\in N^-[u]} h(x) = 2$ all u∈V(D) and hence h is a SETDF on H. Therefore by Lemma 2.4, G has a SEDF

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