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Diffusion in Artificial Kidney (Haemodialyser)

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Abstract

In this paper the diffusion equation is solved by separation of variable method and analyzed Frobenius infinite series approximation. Finally, I have written a computer program using Turbo C++ and simulate the method for several case of interest. I have observed that theoretical findings support the numerical result that I have obtained

Keywords: *Artificial kidney, Diffusion, haemodialysis, Circular cylindrical tube, dialysis, dialysate.*

1 Introduction

An Introduction to Haemodialysis is designed to provide us with information about haemodialysis as a kidney disease treatment option.

Most people are born with two kidneys, each growing to the size of our fist. Your kidneys are bean shaped and are positioned near the middle of our back, on either side of our backbone. Our kidneys are part of the body's urinary system.

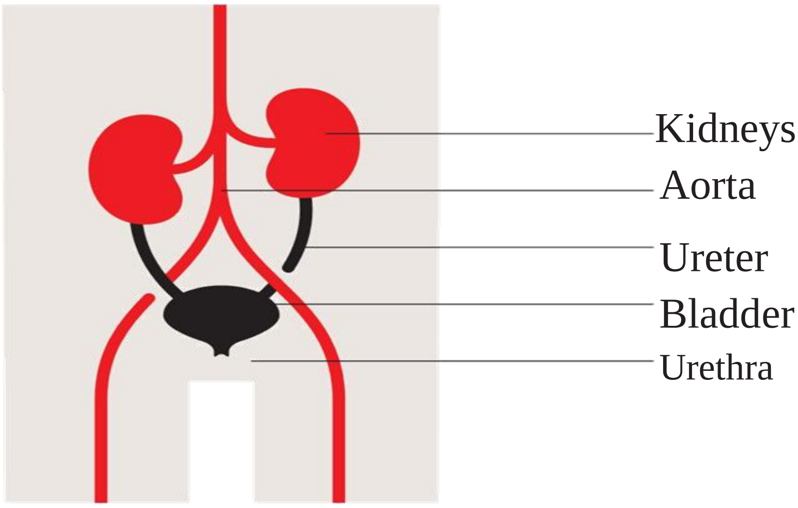

Fig 1 structure of kidney

One of the main functions of kidneys is to maintain the chemical balance of blood by excreting the waste products such as urea, creatine and uric acid in the blood stream. If kidney fails to purify the blood, the impure blood must be purified by external device. The uremia is then removed from it with the help of an external device known as a Haemodialyser, and the purified blood is returned to the body. This process is known as the dialysis of blood.

Sometimes kidney function can change quickly. For example, our kidneys may stop working properly because of a sudden loss of large amounts of blood (e.g., during surgery) or as a result of an accident, illness or infection. A sudden change in kidney function is called acute kidney injury. This is often temporary but can occasionally lead to lasting kidney damage. At this time we need an artificial kidney (hemodialysis).

The first human hemodialysis was performed in a uremic patient by Haas in 1924 at the University of Giessen in Germany. He used a tubular device made of collodion immersed in dialysate solution in a glass cylinder. Haas was able to calculate that the total non-protein nitrogen removed was 2,772 g. He also showed that the presence of some uremic substances in the dialysate and that water could be removed from the blood. In 1928, he first used the anticoagulant, heparin. In 1937, the first flat hemodialysis membrane made of cellophane was produced, which is produced in similar manner to cellulose, but dissolved in alkali and carbon disulfide. The resulting solution is then extruded through a slit and washed multiple times to obtain a transparent semipermeable material.

Willem Kolff from the Netherlands was one of the first investigators interested in the role of toxic solutes in causing the uremic syndrome. In 1940, while taking care of casualties after the German invasion of the Netherlands, his interest in acute renal failure further increased and in 1943 he introduced the rotating drum hemodialysis system using cellophane membranes and an immersion bath and the first recovery of an acute renal failure patient treated with hemodialysis was reported. This was the beginning of what was to become an important clinical reality: artificial renal substitution therapy.

Significant improvements in dialyzer and equipment design occurred during the 1940's and 50's. Nils Alwall developed a new system with a vertical stationary drum kidney and circulating dialysate around the membrane. He was also responsible for applying hydrostatic pressure to achieve ultra filtration. Kolff in turn developed the coil dialyzer using a tubular membrane wrapped around a solid core for use with a single pass dialysis fluid delivery system. This was followed by the twin dialyzer with twin blood pathways, the first disposable hemodialyzer. In 1960, Kiil developed the plate dialyzer that could be reassembled. The system consisted of multiple polypropylene boards supporting flat cellulosic membranes. This parallel flow kidney could be used without a blood pump due to its low resistance.

A new phase in clinical hemodialysis started with the introduction of the Quinton and Scribner AV shunt in 1960. They used silastic tubes fitted with Teflon tips into the radial artery and cephalic vein in the wrist or the posterior tibial artery and saphenous vein at the angle as an arterio-venous shunt. The two tubes ended in expanded couplings to facilitate connection. This shunt provided for the first time continuous circulation of the blood when the patient was not attached to the machine, effectively eliminating clotting and provided ready access for repeated long-term hemodialysis, opening the door to chronic renal replacement therapy.

The next significant advance in vascular access occurred later in the 1960s when Cimino and Brescia first described their native arterio-venous fistula for chronic vascular access. These fistulas are generally created by an end-to-side vein-to-artery anastomosis. A mature native A-V fistula is by far the safest and longest lasting vascular access for hemodialysis.

A Haemodialyser, sometimes called an Artificial Kidney, is regarded as a concentric circular duct (Fig. 2). Blood flows on the inside and outside, the flow called dialyzate which is a solution of some chemicals in water, flows. The wall of the inner cylinder is a semi-permeable membrane that allows urea to diffuse through it. During the process of flow in the duct, blood loses urea which permeates through the duct wall into the dialyzate. By maintaining a continuous supply of fresh dialyzate, the concentration of urea in the dialyzate is maintained lower than that of the blood.

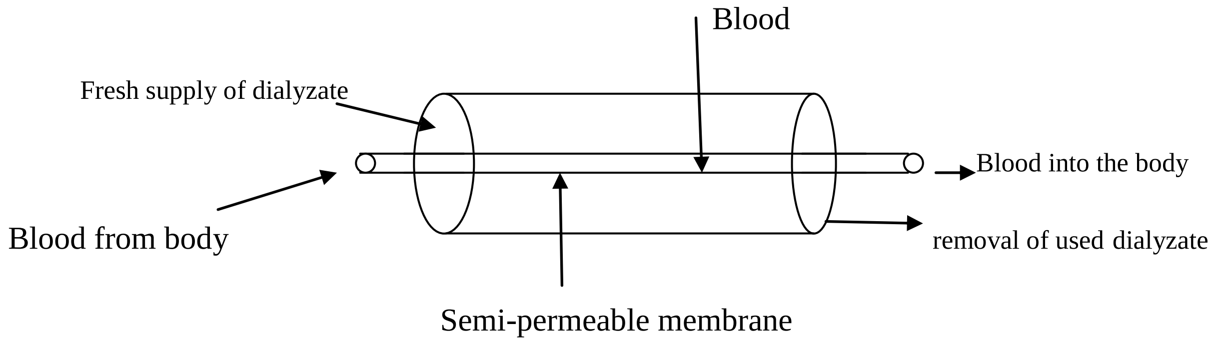

Fig. 2 circular duct Haemodialyser

2 Basic diffusion equations

We have the basic partial differential equation is the diffusion equation given by

i.e. $D\left(\frac{\partial^2 c}{\partial r^2}+\frac{1}{r}\frac{\partial c}{\partial r}+\frac{\partial^2 c}{\partial z^2}\right)=\frac{\partial c}{\partial t}+v\,\frac{\partial c}{\partial z}$ (1)

where c(r, z, t) is the concentration urea in the blood and $v(r,z,t)$ is the velocity of the flow. If we assume steady state ($\frac{\partial c}{\partial t}=0$) laminar flow and maximum velocity in a straight duct, then equation (1) becomes:

$$D\left(\frac{\partial^2 c}{\partial r^2}+\frac{1}{r}\frac{\partial c}{\partial r}+\frac{\partial^2 c}{\partial z^2}\right)=V_m\left(1-\frac{r^2}{R^2}\right)\frac{\partial c}{\partial z} \qquad . \; . \; . \; . \; . \; . \qquad (2)$$

Where c(r, z) is the concentration of urea in the blood at a point (r, z), v_m is the maximum velocity, R is radius of the duct and D is the diffusion coefficient.
If we neglected the longitudinal diffusion term, equation (2) simplified to:

$$D\left(\frac{\partial^2 c}{\partial r^2}+\frac{1}{r}\frac{\partial c}{\partial r}\right)=V_m\left(1-\frac{r^2}{R^2}\right)\frac{\partial c}{\partial z} \qquad . \; . \; . \; . \qquad (3)$$

Now to solve equation (3), we have the following boundary conditions**:**

1. Flow symmetry about the axis and finite concentration at r = 0, which give
   $\frac{\partial c}{\partial r}=0\ at\ r=0$ . . . . (4)

2. Assumption of constant entry concentration, which gives

   $c=c_{in}$ at z = 0 $\quad 0\le r\le R$ . . . . (5)

3. Assumption of constant permeability P and constant c_d in the dialyzate which yields

   $-D\frac{\partial c}{\partial r}=p(c-c_{d})\ at\ r=R, z>0$ . . . . (6)

Introducing the non-dimensional quantities

$$\bar{c}=\frac{c-c_d}{c_{in}-c_d}, \quad \bar{r}=\frac{r}{R}, \quad \bar{z}=\frac{z}{Rp_{e'}}, \quad p_{e'}=\frac{V_m R}{D} \qquad . \; . \; . \qquad (7)$$

Now differentiate $\bar{c}$ with respect to r twice, we have

$$\frac{d\bar{c}}{dr} = \frac{1}{c_{in}-c_d}\frac{dc}{dr} \quad \text{and} \quad \frac{d^2\bar{c}}{dr^2} = \frac{1}{c_{in}-c_d}\frac{d^2c}{dr^2}$$

But $Rd\bar{r} = dr$ and $R^2d\bar{r}^2 = dr^2$

Substitution in the above equation, we get

$$\frac{dc}{dr} = \frac{c_{in}-c_d}{R}\frac{d\bar{c}}{d\bar{r}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

And $$\frac{dc^2}{dr^2} = \frac{c_{in}-c_d}{R^2}\frac{d^2\bar{c}}{d\bar{r}^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

Also differentiate $\bar{c}$ with respect to z, we have

$$\frac{d\bar{c}}{dz} = \frac{1}{c_{in}-c_d}\frac{dc}{dz} \;\Rightarrow\; \frac{dc}{dz} = (c_{in} - c_d)\frac{d\bar{c}}{dz}$$

$$d\bar{z} = \frac{dz}{Rp_{e'}} = \frac{Ddz}{R^2V_m} \quad \Rightarrow dz = \frac{R^2V_m}{D}d\bar{z}$$

Substitution gives

$$\frac{dc}{dz} = \frac{D(c_{in}-c_d)}{R^2V_m}\frac{d\bar{c}}{d\bar{z}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

Now substituting in (8), (9) and (10) in equation (3)

$$D\left(\frac{\partial^2 c}{\partial r^2} + \frac{1}{r}\frac{\partial c}{\partial r}\right) = V_m(1 - \frac{r^2}{R^2})\frac{\partial c}{\partial z}$$

$$D\left(\frac{c_{in}-c_d}{R^2}\frac{d^2\bar{c}}{d\bar{r}^2} + \frac{1}{R\bar{r}}\frac{c_{in}-c_d}{R}\frac{d\bar{c}}{d\bar{r}}\right) = V_m(1 - \bar{r}^2)D\frac{(c_{in}-c_d)}{R^2V_m}\frac{d\bar{c}}{d\bar{z}}$$

$$\Rightarrow \frac{\partial^2\bar{c}}{\partial\bar{r}^2} + \frac{1}{\bar{r}}\frac{\partial\bar{c}}{\partial\bar{r}} = (1 - \bar{r}^2)\frac{\partial\bar{c}}{\partial\bar{z}}$$

From equation (4), at $r = 0$ becomes

$$\frac{dc}{dr} = \frac{c_{in}-c_d}{R}\frac{d\bar{c}}{d\bar{r}} = 0 \Rightarrow \frac{\partial\bar{c}}{\partial\bar{r}} = 0$$

From equation (5), $z = 0$

$$c = \bar{c}(c_{in} - c_d) + c_d = c_{in} \text{ at } z = RP_{e'}\bar{z} = 0, 0 \le R\bar{r} \le R$$

$$\Rightarrow \bar{c}(c_{in} - c_d) = c_{in} - c_d \text{ at } \bar{z} = 0, 0 \le \bar{r} \le 1$$

$$\Rightarrow \bar{c} = 1 \text{ at } \bar{z} = 0, 0 \le \bar{r} \le 1$$

From equation (5), at $r = R$

$$-D\frac{dc}{dr} = -D\frac{c_{in}-c_d}{R}\frac{d\bar{c}}{d\bar{r}} = P(c - c_d) \text{ at } R\bar{r} = R, z = RP_{e'}\bar{z} > 0$$

$$\Rightarrow \frac{d\bar{c}}{d\bar{r}} + \frac{PR}{D}\frac{c-c_d}{c_{in}-c_d} = 0 \text{ at } \bar{r} = 1, \bar{z} > 0$$

Then the system of equation (3) – (6) becomes

$$\frac{\partial^2 \bar{c}}{\partial \bar{r}^2} + \frac{1}{\bar{r}}\frac{\partial \bar{c}}{\partial \bar{r}} = (1-\bar{r}^2)\frac{\partial \bar{c}}{\partial \bar{z}} \quad . \; . \; . \; . \; . \; . \; . \; . \quad (11)$$

$$\bar{c} = 1 \; at \; \bar{z} = 0, 0 \le \bar{r} \le 1, \quad . \; . \; . \; . \; . \; . \; . \quad (12)$$

$$\frac{\partial \bar{c}}{\partial \bar{r}} = 0 \; at \; \bar{r} = 0, \bar{z} > 0 \quad . \; . \; . \; . \; . \; . \quad (13)$$

$$\frac{\partial \bar{c}}{\partial \bar{r}} + sh_w \bar{c} = 0 \; at \; \bar{r} = 1, \bar{z} > 0 \quad . \; . \; . \; . \; . \quad (14)$$

Where $sh_w = \frac{PR}{D}$ is called the Sherwood number.

To solve equation (11) subject to the boundary conditions (12) – (14), we use separation of variables method.

3 Method of separation of variables

We assume a solution of the form $\bar{c}(\bar{r},\bar{z}) = R(\bar{r})Z(\bar{z})$

Substitution in (11) gives

$$\frac{d^2R}{d\bar{r}^2}Z + \frac{1}{\bar{r}}\frac{dR}{d\bar{r}}Z = (1-\bar{r}^2)R\frac{dZ}{d\bar{z}}$$

$$\Rightarrow \frac{d^2R}{Rd\bar{r}^2} + \frac{1}{R\bar{r}}\frac{dR}{d\bar{r}} = \frac{(1-\bar{r}^2)}{Z}\frac{dZ}{d\bar{z}} = -(1-\bar{r}^2)\lambda^2$$

$$\Rightarrow \frac{dZ}{Zd\bar{z}} = -\lambda^2$$

$$\Rightarrow \frac{dZ}{d\bar{z}} + \lambda^2 Z = 0$$

$$\Rightarrow Z(\bar{z}) = e^{-\lambda^2 \bar{z}}$$

Hence the general solution is of the form

$$\bar{c} = \sum_{n=0}^{\infty} A_n R_n(\bar{r}) e^{-\lambda_n^2 \bar{z}} \quad . \; . \; . \; . \; . \; . \; . \; . \quad (15)$$

where $\lambda_n's$ are the eigen values, $R_n's$ are the corresponding eigenfunctions and $A_n's$ are to be determined from the condition of orthogonality of $R_n's$.

Substituting equation (15) in (12) – (14), we have

Then

$$\frac{d^2R_n}{d\bar{r}^2} + \frac{1}{\bar{r}}\frac{dR_n}{d\bar{r}} + \lambda_n^2(1-\bar{r}^2)R_n = 0 \quad . \; . \; . \; . \; . \; . \; . \quad (16)$$

$$\sum_{n=1}^{\infty} A_n R_n(\bar{r}) = 1 \quad . \; . \; . \; . \; . \quad (17)$$

$$\frac{dR_n}{d\bar{r}} = 0 \quad at \; \bar{r} = 0 \quad . \; . \; . \; . \; . \quad (18)$$

$\frac{dR_n}{d\bar{r}} + sh_w R_n = 0$ at $\bar{r} = 1$ . . . . (19)

Assuming Frobenius infinite series solution for (16):

$R_n(\bar{r}) = \sum_{m=o}^{\infty} N_{mn}\, \bar{r}^{m+n}$ . . . . (20)

Substituting (20) in (18) at $\bar{r} = 0$

$$\frac{dR_n}{d\bar{r}} = \sum_{m=0}^{\infty}(m+n)N_{mn}\,\bar{r}^{m+n-1} = \sum_{m=0}^{\infty}(m+n)N_{mn}(0)^{m+n-1} = 0$$

Hence (18) is satisfied and

Now differentiate (19) with respect to $\bar{r}$ twice, we get

$$\frac{dR_n}{d\bar{r}} = \sum_{m=0}^{\infty}(m+n)N_{mn}\bar{r}^{m+n-1} \quad . \quad . \quad . \quad . \quad (21)$$

$$\frac{d^2R_n}{d\bar{r}^2} = \sum_{m=0}^{\infty}(m+n)(m+n-1)\, N_{mn}\bar{r}^{m+n-2} \quad . \quad . \quad (22)$$

Substituting (20), (21) and (22) in (16), gives

$$\sum_{m=0,2,\dots}^{\infty}(m+n)(m+n-1)N_{mn}\,\bar{r}^{m+n-2} + \frac{1}{\bar{r}}\sum_{m=0,2,\dots}^{\infty}(m+n)N_{mn}\bar{r}^{m+n-1}$$
$$+\lambda_n^2(1-\bar{r}^2)\sum_{m=0,2,\dots}^{\infty} N_{mn}\bar{r}^{m+n} = 0$$

Multiplying both sides by $\bar{r}^2$, we get

$$\sum_{m=0}^{\infty}(m+n)(m+n-1)N_{mn}\,\bar{r}^{m+n} + \sum_{m=0}^{\infty}(m+n)N_{mn}\bar{r}^{m+n}$$
$$+\ \lambda_n^2\left(\sum_{m=0}^{\infty} N_{mn}\bar{r}^{m+n+2} - \sum_{m=0}^{\infty} N_{mn}\bar{r}^{m+n+4}\right) = 0$$

$$\sum_{m=0}^{\infty}(m+n)^2\, N_{mn}\,\bar{r}^{m+n} + {\lambda_n}^2\left(\sum_{m=0}^{\infty} N_{mn}\bar{r}^{m+n+2} - \sum_{m=0}^{\infty} N_{mn}\bar{r}^{m+n+4}\right) = 0$$

$$\sum_{m=0}^{\infty}(m+n)^2\, N_{mn}\,\bar{r}^{m+n} + {\lambda_n}^2\left(\sum_{m=2}^{\infty} N_{m-2,n}\bar{r}^{m+n} - \sum_{m=4}^{\infty} N_{m-4,n}\bar{r}^{m+n}\right) = 0$$

Since one of the series in our last equation begins with $m = 4$, we need to separate out the terms corresponding to $m = 0, 1, 2, 3$ and $m = 2, 3$ for the first and second series respectively before combining series:

$$n^2N_{0n}\bar{r}^n + (n+1)^2N_{1n}\bar{r}^{n+1} + (n+2)^2N_{2n}\bar{r}^{n+2} + (n+3)^2N_{3n}\bar{r}^{n+3} + \sum_{m=4}^{\infty}(m+n)^2N_{mn}\bar{r}^{m+n}$$
$$+{\lambda_n}^2(N_{0n}\bar{r}^{n+2} + N_{1n}\bar{r}^{n+3}) + {\lambda_n}^2\sum_{m=4}^{\infty}(N_{m-2,n} - N_{m-4,n})\bar{r}^{m+n} = 0$$

Then

$$n^2N_{0n}\bar{r}^n+(n+1)^2N_{1n}\bar{r}^{n+1}+[(n+2)^2N_{2n}+\lambda_n{}^2N_{0n}]\bar{r}^{n+2}+[(n+3)^2N_{3n}+\lambda_n{}^2N_{1n}]\bar{r}^{n+3}+\sum_{m=4}^{\infty}[(m+n)^2N_{mn}+\lambda_n{}^2(N_{m-2,n}-N_{m-4,n})]\bar{r}^{m+n}=0$$

Dividing out the $\bar{r}^n$ from each term then yields

$$n^2N_{0n}+(n+1)^2N_{1n}\bar{r}^1+[(n+2)^2N_{2n}+\lambda_n{}^2N_{0n}]\bar{r}^2+[(n+3)^2N_{3n}+\lambda_n{}^2N_{1n}]\bar{r}^3+\sum_{m=4}^{\infty}[(m+n)^2N_{mn}+\lambda_n{}^2(N_{m-2,n}-N_{m-4,n})]\bar{r}^m=0$$

We have that each term of power series is equal to zero and by choosing $N_{0n}=1$, we get

i: $n^2N_{0n}=0 \Rightarrow n=0$, since $N_{0n}=1$.

ii: $(n+1)^2N_{1n}=0 \Rightarrow (0+1)^2N_{1n}=0 \Rightarrow N_{1n}=0$

iii: $(n+2)^2N_{2n}+\lambda_n{}^2N_{0n}=0 \Rightarrow (0+2)^2N_{2n}+\lambda_n{}^2N_{0n}=0$

Then $N_{2n}=\frac{-\lambda_n{}^2}{4}$

iv: $(n+3)^2N_{3n}+\lambda_n{}^2N_{1n}=0 \Rightarrow (3)^2N_{3n}+\lambda_n{}^2(0)$

$$N_{3n}=0$$

This indicates that for any odd values of m, $N_{mn}=0$

Thus equation (20) becomes

$$R_n(\bar{r})=\sum_{m=o,2,...}^{\infty}N_{mn}\,\bar{r}^m$$

is the solution, where the summation is taken only even value of m.

v: $(m+n)^2N_{mn}+\lambda_n{}^2(N_{m-2,n}-N_{m-4,n})=0$, But $n=0$

$$N_{mn}=-\frac{\lambda_n{}^2}{m^2}(N_{m-2,n}-N_{m-4,n}),\ \text{For}\ m=4,6,8,... \qquad (23)$$

So we can solve $N_{4n}, N_{6n}, N_{8n}, ...$ from equation (23)

We have the value $N_{0n}=1$ and $N_{2n}=-\frac{\lambda_n{}^2}{4}$

For $m=4$, $N_{4n}=-\frac{\lambda_n{}^2}{16}(N_{2\,n}-N_{0\,n})=-\frac{\lambda_n{}^2}{16}(N_{2\,n}-1)$

$$=\frac{\lambda_n{}^4}{64}+\frac{\lambda_n{}^2}{16}$$

For $m=6$, $N_{6n}=-\frac{\lambda_n{}^2}{36}(N_{4\,n}-N_{2\,n})=-\frac{\lambda_n{}^2}{36}\left(\frac{\lambda_n{}^4}{64}+\frac{\lambda_n{}^2}{16}+\frac{\lambda_n{}^2}{4}\right)$

$$= -\frac{\lambda_n{}^6}{2304} - \frac{5\lambda_n{}^4}{576}$$

For $m = 8$, $N_{8n} = -\frac{\lambda_n{}^2}{64}\left(N_{6\,n} - N_{4\,n}\right) = -\frac{\lambda_n{}^2}{64}\left(-\frac{\lambda_n{}^6}{2304} - \frac{5\lambda_n{}^4}{576} - \frac{\lambda_n{}^4}{64} - \frac{\lambda_n{}^2}{16}\right)$

$$= -\frac{\lambda_n{}^2}{64}\left(-\frac{\lambda_n{}^6}{2304} - \frac{896\lambda_n{}^4}{36864} - \frac{\lambda_n{}^2}{16}\right)$$

$$= \frac{\lambda_n{}^8}{147456} + \frac{14\lambda_n{}^6}{36864} + \frac{\lambda_n{}^4}{1024}$$

Similarly we can solve $N_{10n}, N_{12n}, \ldots$

Now equation (20), becomes

$$R_n(\bar{r}) = 1 - \frac{\lambda_n{}^2}{4}\bar{r}^2 + \left(\frac{\lambda_n{}^4}{64} + \frac{\lambda_n{}^2}{16}\right)\bar{r}^4 - \left(\frac{\lambda_n{}^6}{2304} + \frac{5\lambda_n{}^4}{576}\right)\bar{r}^6 + \left(\frac{\lambda_n{}^8}{147456} + \frac{14\lambda_n{}^6}{36864} + \frac{\lambda_n{}^4}{1024}\right)\bar{r}^8 - \ldots \quad (24)$$

$$R_n(\bar{r}) = 1 - \frac{\lambda_n{}^2}{4}\bar{r}^2 + \left(\frac{\lambda_n{}^4}{64} + \frac{\lambda_n{}^2}{16}\right)\bar{r}^4 + O(\bar{r}^6)$$

Hence by truncating the order six, we have

$$R_n(\bar{r}) = 1 - \frac{\lambda_n{}^2}{4}\bar{r}^2 + \left(\frac{\lambda_n{}^4}{64} + \frac{\lambda_n{}^2}{16}\right)\bar{r}^4 \quad (25)$$

From the boundary condition (19), we can get the value of λ_n by substituting (25)

$\frac{dR_n}{d\bar{r}} + sh_w R_n = 0$, at $\bar{r} = 1$

$$\frac{dR_n}{d\bar{r}} = -\frac{\lambda_n{}^2}{2}\bar{r} + \left(\frac{\lambda_n{}^4}{64} + \frac{\lambda_n{}^2}{16}\right)\bar{r}^3 \quad (26)$$

Then substituting (25) and (26) in (19), we get

$$-\frac{\lambda_n{}^2}{2}\bar{r} + \left(\frac{\lambda_n{}^4}{64} + \frac{\lambda_n{}^2}{16}\right)\bar{r}^3 + sh_w\left(1 - \frac{\lambda_n{}^2}{4}\bar{r}^2 + \left(\frac{\lambda_n{}^4}{64} + \frac{\lambda_n{}^2}{16}\right)\bar{r}^4\right) = 0 \text{ at } \bar{r} = 1$$

$$-\frac{\lambda_n{}^2}{2} + \left(\frac{\lambda_n{}^4}{64} + \frac{\lambda_n{}^2}{16}\right) + sh_w\left(1 - \frac{\lambda_n{}^2}{4}\bar{r}^2 + \left(\frac{\lambda_n{}^4}{64} + \frac{\lambda_n{}^2}{16}\right)\right) = 0$$

$$\lambda_n{}^2\left(1 + \frac{3}{4}sh_w\right) + \lambda_n{}^4\left(\frac{1}{16} + \frac{sh_w}{64}\right) + sh_w = 0$$

$$\lambda_n{}^4(4 + sh_w) - \lambda_n{}^2(16 + 12sh_w) + 64sh_w = 0$$

Let say $\lambda_n{}^2 = k$

$$k^2(4 + sh_w) + k16 + 12sh_w + 64sh_w = 0$$

$$k = \frac{16 + 12sh_w \pm \sqrt{(16 + 12sh_w)^2 - 4 * 64sh_w(4 + sh_w)}}{2 * (4 + sh_w)}$$

$$= \frac{16+12sh_w \pm \sqrt{(256-640sh_w-112sh_w{}^2}}{2*(4+sh_w)}$$

Thus we get four values of λ_n ($i.e. \lambda_0,\ \lambda_1,\ \lambda_2,\ \lambda_3$)

Hence $\lambda_0 = \sqrt{\frac{2(4+3sh_w+\sqrt{16-40sh_w-7sh_w{}^2})}{4+sh_w}}$

$$\lambda_1 = -\sqrt{\frac{2(4+3sh_w+\sqrt{16-40sh_w-7sh_w{}^2})}{4+sh_w}}$$

$$\lambda_2 = \sqrt{\frac{2(4+3sh_w-\sqrt{16-40sh_w-7sh_w{}^2})}{4+sh_w}}$$

$$\lambda_3 = -\sqrt{\frac{2(4+3sh_w-\sqrt{16-40sh_w-7sh_w{}^2})}{4+sh_w}}$$

Hence we have

$$R_0(\bar{r}) = 1 - \frac{\lambda_0{}^2}{4}\bar{r}^2 + \left(\frac{\lambda_0{}^4}{64} + \frac{\lambda_0{}^2}{16}\right)\bar{r}^4$$

$$R_1(\bar{r}) = 1 - \frac{\lambda_1{}^2}{4}\bar{r}^2 + \left(\frac{\lambda_1{}^4}{64} + \frac{\lambda_1{}^2}{16}\right)\bar{r}^4$$

$$R_2(\bar{r}) = 1 - \frac{\lambda_2{}^2}{4}\bar{r}^2 + \left(\frac{\lambda_2{}^4}{64} + \frac{\lambda_2{}^2}{16}\right)\bar{r}^4$$

$$R_3(\bar{r}) = 1 - \frac{\lambda_3{}^2}{4}\bar{r}^2 + \left(\frac{\lambda_3{}^4}{64} + \frac{\lambda_3{}^2}{16}\right)\bar{r}^4$$

From the equation (17), we can get A_n by orthogonality condition of R_n's on the interval [0, 1].

$$A_n = \frac{\int_0^1 R_n(\bar{r})d\bar{r}}{\int_0^1 (R_n(\bar{r})d\bar{r})^2\, d\bar{r}}$$

$$= \frac{\int_0^1 \left(1 - \frac{\lambda_n{}^2}{4}\bar{r}^2 + \left(\frac{\lambda_n{}^4}{64} + \frac{\lambda_n{}^2}{16}\right)\bar{r}^4\right) d\bar{r}}{\int_0^1 (1 - \frac{\lambda_n{}^2}{2}\bar{r}^2 + \left(\frac{3\lambda_n{}^4}{32} + \frac{\lambda_n{}^2}{8}\right)\bar{r}^4 - \frac{\lambda_n{}^2}{2}\left(\frac{\lambda_n{}^4}{64} + \frac{\lambda_n{}^2}{16}\right)\bar{r}^6 + (\frac{\lambda_n{}^4}{64} + \frac{\lambda_n{}^2}{16})^2\bar{r}^8)d\bar{r}}$$

$$= \frac{1 - \frac{\lambda_n^2}{12} + \frac{1}{5}\left(\frac{\lambda_n^4}{64} + \frac{\lambda_n^2}{16}\right)}{1 - \frac{\lambda_n^2}{6} + \frac{1}{5}\left(\frac{3\lambda_n^4}{32} + \frac{\lambda_n^2}{8}\right) - \frac{\lambda_n^2}{14}\left(\frac{\lambda_n^4}{64} + \frac{\lambda_n^2}{16}\right) + \frac{1}{9}\left(\frac{\lambda_n^4}{64} + \frac{\lambda_n^2}{16}\right)^2}$$

Then substituting the values of A_n and R_n into (15), we get

$$\bar{c} = \sum_{n=0}^{3} A_n\, R_n \exp\left(-\lambda_n^2 \bar{z}\right)$$

$$= A_0 R_0 \exp\left(-\lambda_0^2 \bar{z}\right) + A_1 R_1 \exp\left(-\lambda_1^2 \bar{z}\right) + A_2 R_2 \exp\left(-\lambda_2^2 \bar{z}\right) + A_3 R_3 \exp\left(-\lambda_3^2 \bar{z}\right)$$

4 Result and Discussion

The main result of this study is to know the concentration distribution over the Artificial kidney. To get such value of concentration along the cylinder, we fix different values of the value sherwood number, sh_w.

By using c++ code for our calculation we get the following values represented graphically.

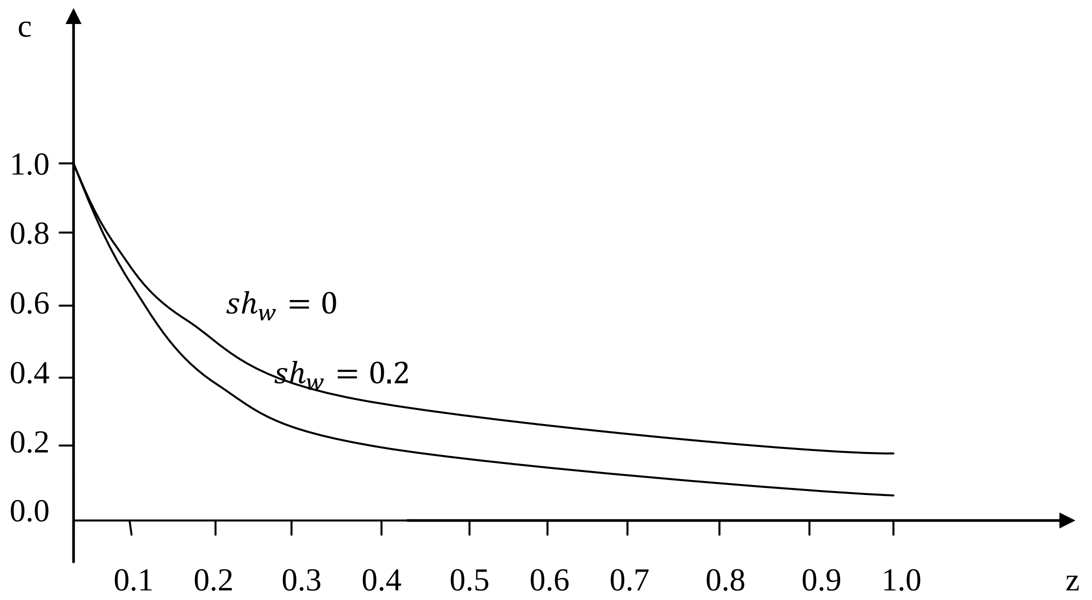

Fig 3.2 concentration for different values of sh_w

For different values of Sherwood number, we get different concentration. As we can from above graph as Sherwood number, sh_w, increases the concentration decreases and the concentration decreases when the fluid goes from the entrance to the exit.

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