IJETST- Vol.||04||Issue||06||Pages 5233-5241||June||ISSN 2348-9480 2017

Open access Journal International Journal of Emerging Trends in Science and Technology
IC Value: 76.89 (Index Copernicus) Impact Factor: 4.219 DOI: https://dx.doi.org/10.18535/ijetst/v4i6.03

An Another Generalized Fibonacci Sequence

Author
Guesh Gebrezgiher Gebrekirstos
Department of Mathematics, College of Natural and Computational science
Adigrat University, Adigrat, Ethiopia
Email: *guesh2002@gmail.com*

ABSTRACT
In this paper, we establish a generalized Fibonacci sequence defined by $S_n = S_{n-1} + S_{n-2} + S_{n-3} + S_{n-4} + S_{n-5} + S_{n-6} + S_{n-7} + S_{n-8}$. *Some properties of sequence (S_n) are also established.*

1. INTRODUCTION

The Fibonacci Sequence (F_n) is defined as $F_n = F_{n-1} + F_{n-2}$ and $F_0 = 1, F_1 = 1$. This Sequence is generalized in different ways.[1] (1.1)

Modifying the recurrence relation such that each term is the sum of the three preceding terms, [2]

$$P_n = P_{n-1} + P_{n-2} + P_{n-3} \; (n \geq 3) \tag{1.2}$$

Where P_0, P_1, P_2 are arbitrary algebraic integers all of which are not zero

Modifying the recurrence relation such that each term is the sum of the four preceding terms, [3]

$$Q_n = Q_{n-1} + Q_{n-2} + Q_{n-3} + Q_{n-4} \; (n \geq 4) \tag{1.3}$$

Where Q_0, Q_1, Q_2, Q_3 are arbitrary algebraic integers all of which are not zero

Modifying the recurrence relation such that each term is the sum of the five preceding terms, [4]

$$D_n = D_{n-1} + D_{n-2} + D_{n-3} + D_{n-4} + D_{n-5} \; (n \geq 5) \tag{1.4}$$

Where D_0, D_1, D_2, D_3, D_4 are arbitrary algebraic integers all of which are not zero

Modifying the recurrence relation such that each term is the sum of the six preceding terms, [4]

$$X_n = X_{n-1} + X_{n-2} + X_{n-3} + X_{n-4} + X_{n-5} + X_{n-6} \; (n \geq 6) \tag{1.5}$$

Where X_0, X_1, X_2, X_3, X_4, X_5 are arbitrary algebraic integers all of which are not zero

Modifying the recurrence relation such that each term is the sum of the seven preceding terms, [4]

$$Y_n = Y_{n-1} + Y_{n-2} + Y_{n-3} + Y_{n-4} + Y_{n-5} + Y_{n-6} + Y_{n-7} \; (n \geq 7) \tag{1.6}$$

Where Y_0, Y_1, Y_2, Y_3, Y_4, Y_5, Y_6 are arbitrary algebraic integers all of which are not zero

2. The Generalized Sequence (S_n)

We consider the Sequence:

$(S_n) = S_0, S_1, S_2$,-------,S_n,--------

Where $S_0, S_1, S_2, S_3, S_4, S_5, S_6, S_7$ are arbitrary algebraic integers all of which are not zero, and

$$S_n = S_{n-1} + S_{n-2} + S_{n-3} + S_{n-4} + S_{n-5} + S_{n-6} + S_{n-7} + S_{n-8} \ (n \geq 8) \quad (2.1)$$

We also consider the Sequence

$(A_n) = A_0, A_1, A_2, A_3$,--------

Where

$A_0 = S_2 - S_1 - S_0$, $A_1 = S_3 – S_2 - S_1$, $A_2 = S_4 – S_3 - S_2$, $A_3 = S_5 – S_4 - S_3$,

$A_4 = S_6 – S_5 - S_4$, $A_5 = S_7 – S_6 - S_5$, $A_6 = S_8 – S_7 - S_6$ (2.2)

With

$$A_n = S_{n-1} + S_{n-2} + S_{n-3} + S_{n-4} + S_{n-5} + S_{n-6} + S_{n-7} \ (n \geq 7) \quad (2.3)$$

and

$(B_n) = B_0, B_1, B_2, B_3$,--------

Where

$B_0 = S_3 - S2 - S_1 - S_0$, $B_1 = S_4 - S_3 – S_2 - S_1$, $B_2 = S_5 - S_4 – S_3 - S_2$, $B_3 = S_6 - S_5 – S_4 - S_3$

$B_4 = S_7 - S_6 – S_5 - S_4$, $B_5 = S_8 - S_7 – S_6 - S_5$ (2.4)

With

$$B_n = S_{n-1} + S_{n-2} + S_{n-3} + S_{n-4} + S_{n-5} + S_{n-6} \ (n \geq 6) \quad (2.5)$$

And

$(C_n) = C_0, C_1, C_2, C_3$,--------

Where

$C_0 = S_4 - S_3 - S2 - S_1 - S_0$, $C_1 = S_5 - S_4 - S_3 – S_2 - S_1$, $C_2 = S_6 - S_5 - S_4 – S_3 - S_2$

$C_3 = S_7 - S_6 - S_5 – S_4 - S_3$, $C_4 = S_8 - S_7 - S_6 – S_5 - S_4$ (2.6)

With

$$C_n = S_{n-1} + S_{n-2} + S_{n-3} + S_{n-4} + S_{n-5} \ (n \geq 5) \quad (2.7)$$

and

$(D_n) = D_0, D_1, D_2, D_3$,--------

Where

$D_0 = S_5 - S_4 - S_3 - S2 - S_1 - S_0$, $D_1 = S_6 - S_5 - S_4 - S_3 – S_2 - S_1$,

$D_2 = S_7 - S_6 - S_5 - S_4 – S_3 - S_2$, $D_3 = S_8 - S_7 - S_6 - S_5 – S_4 - S_3$ (2.8)

With

$$D_n = S_{n-1} + S_{n-2} + S_{n-3} + S_{n-4} \ (n \geq 4) \quad (2.9)$$

and

$(E_n) = E_0, E_1, E_2, E_3$,--------

Where

$E_0 = S_6 - S_5 - S_4 - S_3 - S2 - S_1 - S_0$, $E_1 = S_7 – S_6 - S_5 - S_4 - S_3 – S_2 - S_1$

$E_2 = S_8 – S_7 - S_6 - S_5 - S_4 – S_3 - S_2$ (2.10)

With

$$E_n = S_{n-1} + S_{n-2} + S_{n-3} \ (n \geq 3) \quad (2.11)$$

and

$(F_n) = F_0, F_1, F_2, F_3$,--------

Where

$F_0 = S_7 - S_6 - S_5 - S_4 - S_3 - S2 - S_1 - S_0$, $F_1 = S_8 - S_7 – S_6 - S_5 - S_4 - S_3 – S_2 - S_1$ (2.12)

With

$$F_n = S_{n-1} + S_{n-2} \ (n\geq 2) \qquad (2.13)$$

From (2.3) and (2.1), we have for n≥15

$$\begin{aligned} A_n = \ & S_{n-9}+S_{n-8}+S_{n-7}+S_{n-6}+S_{n-5}+S_{n-4}+S_{n-3}+S_{n-2} \\ & +S_{n-10}+S_{n-9}+S_{n-8}+S_{n-7}+S_{n-6}+S_{n-5}+S_{n-4}+S_{n-3} \\ & +S_{n-11}+S_{n-10}+S_{n-9}+S_{n-8}+S_{n-7}+S_{n-6}+S_{n-5}+S_{n-4} \\ & +S_{n-12}+S_{n-11}+S_{n-10}+S_{n-9}+S_{n-8}+S_{n-7}+S_{n-6}+S_{n-5} \\ & +S_{n-13}+S_{n-12}+S_{n-11}+S_{n-10}+S_{n-9}+S_{n-8}+S_{n-7}+S_{n-6} \\ & +S_{n-14}+S_{n-13}+S_{n-12}+S_{n-11}+S_{n-10}+S_{n-9}+S_{n-8}+S_{n-7} \\ & +S_{n-15}+S_{n-14}+S_{n-13}+S_{n-12}+S_{n-11}+S_{n-10}+S_{n-9}+S_{n-8} \end{aligned}$$

$$A_n = A_{n-1}+A_{n-2}+A_{n-3}+A_{n-4}+A_{n-5}+A_{n-6}+A_{n-7}+A_{n-8}$$

Using (2.3) and (2.2) we obtain

$$\begin{aligned} A_{14} = \ & S_{12}+S_{11}+S_{10}+S_9+S_8+S_7+S_6+S_5 \\ & +S_{11}+S_{10}+S_9+S_8+S_7+S_6+S_5+S_4 \\ & +S_{10}+S_9+S_8+S_7+S_6+S_5+S_4+S_3 \\ & +S_9+S_8+S_7+S_6+S_5+S_4+S_3+S_2 \\ & +S_8+S_7+S_6+S_5+S_4+S_3+S_2+S_1 \\ & +S_7+S_6+S_5+S_4+S_3+S_2+S_1+S_0 \\ & +S_6+S_5+S_4+S_3+S_2+S_1+S_0 \end{aligned}$$

And Similarly,

$$A_{13} = A_{12}+A_{11}+A_{10}+A_9+A_8+A_7+A_6+A_5$$

$$A_{12} = A_{11}+A_{10}+A_9+A_8+A_7+A_6+A_5+A_4$$

$$A_{11} = A_{10}+A_9+A_8+A_7+A_6+A_5+A_4+A_3$$

Hence we have for n≥8

$$A_n = A_{n-1}+A_{n-2}+A_{n-3}+A_{n-4}+A_{n-5}+A_{n-6}+A_{n-7}+A_{n-8} \qquad (2.14)$$

Proceeding on similar lines, it can be easily shown that for n≥8

$$\begin{aligned} B_n = \ & S_{n-9}+S_{n-8}+S_{n-7}+S_{n-6}+S_{n-5}+S_{n-4}+S_{n-3}+S_{n-2} \\ & +S_{n-10}+S_{n-9}+S_{n-8}+S_{n-7}+S_{n-6}+S_{n-5}+S_{n-4}+S_{n-3} \\ & +S_{n-11}+S_{n-10}+S_{n-9}+S_{n-8}+S_{n-7}+S_{n-6}+S_{n-5}+S_{n-4} \\ & +S_{n-12}+S_{n-11}+S_{n-10}+S_{n-9}+S_{n-8}+S_{n-7}+S_{n-6}+S_{n-5} \\ & +S_{n-13}+S_{n-12}+S_{n-11}+S_{n-10}+S_{n-9}+S_{n-8}+S_{n-7}+S_{n-6} \\ & +S_{n-14}+S_{n-13}+S_{n-12}+S_{n-11}+S_{n-10}+S_{n-9}+S_{n-8}+S_{n-7} \end{aligned}$$

Hence we have for n≥8

$$B_n = B_{n-1} + B_{n-2} + B_{n-3} + B_{n-4} + B_{n-5} + B_{n-6} + B_{n-7} + B_{n-8} \tag{2.15}$$

Proceeding on similar lines, it can easily shown that, for n≥8

$$\begin{aligned} C_n = {} & S_{n-9} + S_{n-8} + S_{n-7} + S_{n-6} + S_{n-5} + S_{n-4} + S_{n-3} + S_{n-2} \\ & + S_{n-10} + S_{n-9} + S_{n-8} + S_{n-7} + S_{n-6} + S_{n-5} + S_{n-4} + S_{n-3} \\ & + S_{n-11} + S_{n-10} + S_{n-9} + S_{n-8} + S_{n-7} + S_{n-6} + S_{n-5} + S_{n-4} \\ & + S_{n-12} + S_{n-11} + S_{n-10} + S_{n-9} + S_{n-8} + S_{n-7} + S_{n-6} + S_{n-5} \\ & + S_{n-13} + S_{n-12} + S_{n-11} + S_{n-10} + S_{n-9} + S_{n-8} + S_{n-7} + S_{n-6} \end{aligned}$$

Hence we have for n≥8

$$C_n = C_{n-1} + C_{n-2} + C_{n-3} + C_{n-4} + C_{n-5} + C_{n-6} + C_{n-7} + C_{n-8} \tag{2.16}$$

Proceeding on similar lines, it can easily shown that, for n≥8

$$\begin{aligned} D_n = {} & S_{n-9} + S_{n-8} + S_{n-7} + S_{n-6} + S_{n-5} + S_{n-4} + S_{n-3} + S_{n-2} \\ & + S_{n-10} + S_{n-9} + S_{n-8} + S_{n-7} + S_{n-6} + S_{n-5} + S_{n-4} + S_{n-3} \\ & + S_{n-11} + S_{n-10} + S_{n-9} + S_{n-8} + S_{n-7} + S_{n-6} + S_{n-5} + S_{n-4} \\ & + S_{n-12} + S_{n-11} + S_{n-10} + S_{n-9} + S_{n-8} + S_{n-7} + S_{n-6} + S_{n-5} \end{aligned}$$

Hence we have for n≥8

$$D_n = D_{n-1} + D_{n-2} + D_{n-3} + D_{n-4} + D_{n-5} + D_{n-6} + D_{n-7} + D_{n-8} \tag{2.17}$$

Proceeding on similar lines,it can easily shown that, for n≥8

$$\begin{aligned} E_n = {} & S_{n-9} + S_{n-8} + S_{n-7} + S_{n-6} + S_{n-5} + S_{n-4} + S_{n-3} + S_{n-2} \\ & + S_{n-10} + S_{n-9} + S_{n-8} + S_{n-7} + S_{n-6} + S_{n-5} + S_{n-4} + S_{n-3} \\ & + S_{n-11} + S_{n-10} + S_{n-9} + S_{n-8} + S_{n-7} + S_{n-6} + S_{n-5} + S_{n-4} \end{aligned}$$

Hence we have for n≥8

$$E_n = E_{n-1} + E_{n-2} + E_{n-3} + E_{n-4} + E_{n-5} + E_{n-6} + E_{n-7} + E_{n-8} \tag{2.18}$$

Proceeding on similar lines, it can easily shown that, for n≥8

$$\begin{aligned} F_n = {} & S_{n-9} + S_{n-8} + S_{n-7} + S_{n-6} + S_{n-5} + S_{n-4} + S_{n-3} + S_{n-2} \\ & + S_{n-10} + S_{n-9} + S_{n-8} + S_{n-7} + S_{n-6} + S_{n-5} + S_{n-4} + S_{n-3} \end{aligned}$$

Hence we have for n≥8

$$F_n = F_{n-1} + F_{n-2} + F_{n-3} + F_{n-4} + F_{n-5} + F_{n-6} + F_{n-7} + F_{n-8} \tag{2.19}$$

Thus, the Sequences (A_n) , (B_n) , (C_n), (D_n), (E_n) and (F_n) are the special cases of (S_n) by giving different initial values.
On taking

$S_0 = S_1 = S_2 = S_3 = S_4 = 0;\ S_5 = S_6 = 1;\ S_7 = 2$
$S_0 = S_1 = S_2 = S_3 = 0; S_4 = 1;\ S_5 = 0;\ S_6 = 1;\ S_7 = 2$
$S_0 = S_1 = S_2 = 0;\ S_3 = 1;\ S_4 = S_5 = 0;\ S_6 = 1;\ S_7 = 2$
$S_0 = S_1 = 0;\ S_2 = 1;\ S_3 = S_4 = S_5 = 0;\ S_6 = 1;\ S_7 = 2$
$S_0 = 0;\ S_1 = 1;\ S_2 = S_3 = S_4 = S_5 = 0;\ S_6 = 1;\ S_7 = 2$
$S_0 = 1;\ S_1 = S_2 = S_3 = S_4 = S_5 = 0;\ S_6 = 1;\ S_7 = 2$
$S_0 = S_1 = S_2 = S_3 = S_4 = S_5 = 0;\ S_6 = 1;\ S_7 = 2$

0,0,0,0,0,1,1,2,4,8,16,32,64,128,----------,S_n,-----------
0,0,0,0,1,0,1,2,4,8,16,32,64,127,----------,A_n,----------
0,0,0,1,0,0,1,2,4,8,16,32,63,126,----------,B_n,----------
0,0,1,0,0,0,1,2,4,8,16,31,62,124,----------,C_n,----------
0,1,0,0,0,0,1,2,4,8,15,30,60,120,----------,D_n,----------
1,0,0,0,0,0,1,2,4,7,14,28,58,112,----------,E_n,----------
0,0,0,0,0,0,1,2,3,6,12,24,48,96,-----------,F_n,----------

Hence we find that

$$A_n = S_{n-1} + S_{n-2} + S_{n-3} + S_{n-4} + S_{n-5} + S_{n-6} + S_{n-7}$$
$$B_n = S_{n-1} + S_{n-2} + S_{n-3} + S_{n-4} + S_{n-5} + S_{n-6}$$
$$C_n = S_{n-1} + S_{n-2} + S_{n-3} + S_{n-4} + S_{n-5}$$
$$D_n = S_{n-1} + S_{n-2} + S_{n-3} + S_{n-4}$$
$$E_n = S_{n-1} + S_{n-2} + S_{n-3}$$
$$F_n = S_{n-1} + S_{n-2}$$

3. LINEAR SUMS AND SOME PROPERTIES

We have derived simple properties of the sequences (S_n), (A_n) , (B_n) , (C_n), (D_n), (E_n) and (F_n), expressing each of the terms S_8 ,S_9 , S_{10} , ----------- S_{n+7} as the sum of its eight preceding terms as given in (2.1). Adding both sides we obtain on simplification:

$$\sum_{i=0}^{n} S_i = \frac{1}{7}\left[S_{n+7} - S_{n+5} - 2S_{n+4} - 3S_{n+3} - 4S_{n+2} - 5S_{n+1} + S_n - \left(S_7 - S_5 - 2S_4 - 3S_3 - 4S_2 - 5S_1 - 6S_0\right)\right] \tag{3.1}$$

On using (2.1), (2.2), (2.4), (2.6), (2.8), (2.10) and (2.12), we get

$$\sum_{i=0}^{n} S_{8i} = \sum_{i=0}^{8n-1} S_i + S_0 \tag{3.2}$$

$$\sum_{i=0}^{n} S_{8i+2} = \sum_{i=0}^{8n+1} S_i + A_0 \tag{3.3}$$

$$\sum_{i=0}^{n} S_{8i+3} = \sum_{i=0}^{8n+2} S_i + B_0 \quad (3.4)$$

$$\sum_{i=0}^{n} S_{8i+4} = \sum_{i=0}^{8n+3} S_i + C_0 \quad (3.5)$$

$$\sum_{i=0}^{n} S_{8i+5} = \sum_{i=0}^{8n+4} S_i + D_0 \quad (3.6)$$

$$\sum_{i=0}^{n} S_{8i+6} = \sum_{i=0}^{8n+5} S_i + E_0 \quad (3.7)$$

$$\sum_{i=0}^{n} S_{8i+7} = \sum_{i=0}^{8n+6} S_i + F_0 \quad (3.8)$$

$$\sum_{i=0}^{n} S_{8i+8} = \sum_{i=0}^{8n+7} S_i + (F_1 - S_0) \quad (3.9)$$

$$\sum_{i=0}^{n} S_{8i+7} = \sum_{i=0}^{8n+6} S_i + (E_1 - S_0) \quad (3.10)$$

$$\sum_{i=0}^{n} S_{8i+6} = \sum_{i=0}^{8n+5} S_i + (D_1 - S_0) \quad (3.11)$$

$$\sum_{i=0}^{n} S_{8i+5} = \sum_{i=0}^{8n+4} S_i + (C_1 - S_0) \quad (3.12)$$

$$\sum_{i=0}^{n} S_{8i+4} = \sum_{i=0}^{8n+3} S_i + (B_1 - S_0) \quad (3.13)$$

$$\sum_{i=0}^{n} S_{8i+3} = \sum_{i=0}^{8n+2} S_i + (A_1 - S_0) \quad (3.14)$$

On using (2.1), (2.3), (2.5), (2.7), (2.9) and (2.11) we get,

$$\sum_{i=1}^{48n} F_{2i} = \sum_{i=1}^{32n} E_{3i} = \sum_{i=1}^{24n} D_{4i} = \sum_{i=1}^{16n} C_{6i} = \sum_{i=1}^{12n} B_{8i} = \sum_{i=1}^{8n} A_{12i} = \sum_{i=0}^{96n-1} S_i \quad (3.15)$$

4. PROPERTY OF SEQUENCE (S_n)

Theorem for Sequence (S_n):

$$\begin{vmatrix} S_n & S_{n+1} & S_{n+2} & S_{n+3} & S_{n+4} & S_{n+5} & S_{n+6} & S_{n+7} \\ S_{n+1} & S_{n+2} & S_{n+3} & S_{n+4} & S_{n+5} & S_{n+6} & S_{n+7} & S_{n+8} \\ S_{n+2} & S_{n+3} & S_{n+4} & S_{n+5} & S_{n+6} & S_{n+7} & S_{n+8} & S_{n+9} \\ S_{n+3} & S_{n+4} & S_{n+5} & S_{n+6} & S_{n+7} & S_{n+8} & S_{n+9} & S_{n+10} \\ S_{n+4} & S_{n+5} & S_{n+6} & S_{n+7} & S_{n+8} & S_{n+9} & S_{n+10} & S_{n+11} \\ S_{n+5} & S_{n+6} & S_{n+7} & S_{n+8} & S_{n+9} & S_{n+10} & S_{n+11} & S_{n+12} \\ S_{n+6} & S_{n+7} & S_{n+8} & S_{n+9} & S_{n+10} & S_{n+11} & S_{n+12} & S_{n+13} \\ S_{n+7} & S_{n+8} & S_{n+9} & S_{n+10} & S_{n+11} & S_{n+12} & S_{n+13} & S_{n+14} \end{vmatrix} = (-1)^{n+1}$$

Proof: Consider the determinant

$$\Delta = \begin{vmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{vmatrix}$$

The value of determinant is +1; we have

$$\Delta^2 = \begin{vmatrix} 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{vmatrix}$$

Hence by Mathematical Induction,

$$\Delta^n = \begin{vmatrix} S_{n+1} & A_{n+1} & B_{n+1} & C_{n+1} & D_{n+1} & E_{n+1} & F_{n+1} & S_n \\ S_n & A_n & B_n & C_n & D_n & E_n & F_n & S_{n-1} \\ S_{n-1} & A_{n-1} & B_{n-1} & C_{n-1} & D_{n-1} & E_{n-1} & F_{n-1} & S_{n-2} \\ S_{n-2} & A_{n-2} & B_{n-2} & C_{n-2} & D_{n-2} & E_{n-2} & F_{n-2} & S_{n-3} \\ S_{n-3} & A_{n-3} & B_{n-3} & C_{n-3} & D_{n-3} & E_{n-3} & F_{n-3} & S_{n-4} \\ S_{n-4} & A_{n-4} & B_{n-4} & C_{n-4} & D_{n-4} & E_{n-4} & F_{n-4} & S_{n-5} \\ S_{n-5} & A_{n-5} & B_{n-5} & C_{n-5} & D_{n-5} & E_{n-5} & F_{n-5} & S_{n-6} \\ S_{n-6} & A_{n-6} & B_{n-6} & C_{n-6} & D_{n-6} & E_{n-6} & F_{n-6} & S_{n-7} \end{vmatrix}$$

Writing $F_{n+1} = S_n + S_{n-1}$, the R.H.S can be written as the sum of the two determinants and of which is obviously zaero. Therefore,

$$\Delta^n = \begin{vmatrix} S_{n+1} & A_{n+1} & B_{n+1} & C_{n+1} & D_{n+1} & E_{n+1} & S_{n-1} & S_n \\ S_n & A_n & B_n & C_n & D_n & E_n & S_{n-2} & S_{n-1} \\ S_{n-1} & A_{n-1} & B_{n-1} & C_{n-1} & D_{n-1} & E_{n-1} & S_{n-3} & S_{n-2} \\ S_{n-2} & A_{n-2} & B_{n-2} & C_{n-2} & D_{n-2} & E_{n-2} & S_{n-4} & S_{n-3} \\ S_{n-3} & A_{n-3} & B_{n-3} & C_{n-3} & D_{n-3} & E_{n-3} & S_{n-5} & S_{n-4} \\ S_{n-4} & A_{n-4} & B_{n-4} & C_{n-4} & D_{n-4} & E_{n-4} & S_{n-6} & S_{n-5} \\ S_{n-5} & A_{n-5} & B_{n-5} & C_{n-5} & D_{n-5} & E_{n-5} & S_{n-7} & S_{n-6} \\ S_{n-6} & A_{n-6} & B_{n-6} & C_{n-6} & D_{n-6} & E_{n-6} & S_{n-8} & S_{n-7} \end{vmatrix}$$

Writing $E_{n+1} = S_n + S_{n-1} + S_{n-2}$ the R.H.S can be written as sum of three determinants two of which are zero, therefore

$$\Delta^n = \begin{vmatrix} S_{n+1} & A_{n+1} & B_{n+1} & C_{n+1} & D_{n+1} & S_{n-2} & S_{n-1} & S_n \\ S_n & A_n & B_n & C_n & D_n & S_{n-3} & S_{n-2} & S_{n-1} \\ S_{n-1} & A_{n-1} & B_{n-1} & C_{n-1} & D_{n-1} & S_{n-4} & S_{n-3} & S_{n-2} \\ S_{n-2} & A_{n-2} & B_{n-2} & C_{n-2} & D_{n-2} & S_{n-5} & S_{n-4} & S_{n-3} \\ S_{n-3} & A_{n-3} & B_{n-3} & C_{n-3} & D_{n-3} & S_{n-6} & S_{n-5} & S_{n-4} \\ S_{n-4} & A_{n-4} & B_{n-4} & C_{n-4} & D_{n-4} & S_{n-7} & S_{n-6} & S_{n-5} \\ S_{n-5} & A_{n-5} & B_{n-5} & C_{n-5} & D_{n-5} & S_{n-8} & S_{n-7} & S_{n-6} \\ S_{n-6} & A_{n-6} & B_{n-6} & C_{n-6} & D_{n-6} & S_{n-9} & S_{n-8} & S_{n-7} \end{vmatrix}$$

on writing,

$D_{n+1} = S_n + S_{n-1} + S_{n-2} + S_{n-4}$
$C_{n+1} = S_n + S_{n-1} + S_{n-2} + S_{n-3} + S_{n-4}$
$B_{n+1} = S_n + S_{n-1} + S_{n-2} + S_{n-3} + S_{n-4} + S_{n-5}$
$A_{n+1} = S_n + S_{n-1} + S_{n-2} + S_{n-3} + S_{n-4} + S_{n-5} + S_{n-6}$

We have,

$$\Delta^n = \begin{vmatrix} S_{n+1} & S_{n-6} & S_{n-5} & S_{n-4} & S_{n-3} & S_{n-2} & S_{n-1} & S_n \\ S_n & S_{n-7} & S_{n-6} & S_{n-5} & S_{n-4} & S_{n-3} & S_{n-2} & S_{n-1} \\ S_{n-1} & S_{n-8} & S_{n-7} & S_{n-6} & S_{n-5} & S_{n-4} & S_{n-3} & S_{n-2} \\ S_{n-2} & S_{n-9} & S_{n-8} & S_{n-7} & S_{n-6} & S_{n-5} & S_{n-4} & S_{n-3} \\ S_{n-3} & S_{n-10} & S_{n-9} & S_{n-8} & S_{n-7} & S_{n-6} & S_{n-5} & S_{n-4} \\ S_{n-4} & S_{n-11} & S_{n-10} & s_{n-9} & S_{n-8} & S_{n-7} & S_{n-6} & S_{n-5} \\ S_{n-5} & S_{n-12} & S_{n-11} & S_{n-10} & S_{n-9} & S_{n-8} & S_{n-7} & S_{n-6} \\ S_{n-6} & S_{n-13} & S_{n-12} & S_{n-11} & S_{n-10} & S_{n-9} & S_{n-8} & S_{n-7} \end{vmatrix}$$

On Rearranging we get,

$$\Delta^n = \begin{vmatrix} S_{n+1} & S_n & S_{n-1} & S_{n-2} & S_{n-3} & S_{n-4} & S_{n-5} & S_{n-6} \\ S_n & S_{n-1} & S_{n-2} & S_{n-3} & S_{n-4} & S_{n-5} & S_{n-6} & S_{n-7} \\ S_{n-1} & S_{n-2} & S_{n-3} & S_{n-4} & S_{n-5} & S_{n-6} & S_{n-7} & S_{n-8} \\ S_{n-2} & S_{n-3} & S_{n-4} & S_{n-5} & S_{n-6} & S_{n-7} & S_{n-8} & S_{n-9} \\ S_{n-3} & S_{n-4} & S_{n-5} & S_{n-6} & S_{n-7} & S_{n-8} & S_{n-9} & S_{n-10} \\ S_{n-4} & S_{n-5} & S_{n-6} & S_{n-7} & S_{n-8} & S_{n-9} & S_{n-10} & S_{n-11} \\ S_{n-5} & S_{n-6} & S_{n-7} & S_{n-8} & S_{n-9} & S_{n-10} & S_{n-11} & S_{n-12} \\ S_{n-6} & S_{n-7} & S_{n-8} & S_{n-9} & S_{n-10} & S_{n-11} & S_{n-12} & S_{n-13} \end{vmatrix}$$

On putting n -13 = m, we get

$$\begin{vmatrix} S_{m+14} & S_{m+13} & S_{m+12} & S_{m+11} & S_{m+10} & S_{m+9} & S_{m+8} & S_{m+7} \\ S_{m+13} & S_{m+12} & S_{m+11} & S_{m+10} & S_{m+9} & S_{m+8} & S_{m+7} & S_{m+6} \\ S_{m+12} & S_{m+11} & S_{m+10} & S_{m+9} & S_{m+8} & S_{m+7} & S_{m+6} & S_{m+5} \\ S_{m+11} & S_{m+10} & S_{m+9} & S_{m+8} & S_{m+7} & S_{m+6} & S_{m+5} & S_{m+4} \\ S_{m+10} & S_{m+9} & S_{m+8} & S_{m+7} & S_{m+6} & S_{m+5} & S_{m+4} & S_{m+3} \\ S_{m+9} & S_{m+8} & S_{m+7} & S_{m+6} & S_{m+5} & S_{m+4} & S_{m+3} & S_{m+2} \\ S_{m+8} & S_{m+7} & S_{m+6} & S_{m+5} & S_{m+4} & S_{m+3} & S_{m+2} & S_{m+1} \\ S_{m+7} & S_{m+6} & S_{m+5} & S_{m+4} & S_{m+3} & S_{m+2} & S_{m+1} & S_{m} \end{vmatrix} = (-1)^{m+12}$$

Rearranging the determinant and replacing m by n, we get the result.

References

1. Brother, v. Alfred., An Introduction to Fibonacci discovery, The Fibonacci Association, Calif.,(1965)
2. Waddill and lovis sacks., Fibonacci Quarterly 5(3), (1967), 209-222
3. Jaiswal D.V., LABDEV Jour. Of sequences and technology(India), 7(2), (1969), 67-71
4. Saijay Harne and C.L Parihar., Matrix Properties of generalized Fibonacci sequence,J.Indian Acad.Math.Vol.17, No.2, 1995, 171-177