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Note on lexicographical ordering

Authors

Cheng-Tan Tung[1] , Shu-Cheng Lin[2] , Peterson Julian[3]

[1] Department of Information Management, Central Police University

[2] Department of Hospital Management, Lee-Ming Institute of Technology

[3] Department of Traffic Science, Central Police University

Email: *petersonjulian44328@gmail.com*

Abstract

Farhadinia (2016) developed a lexicographical ordering with respect to hesitant fuzzy elements. In this article, we will demonstrate that his novel approach fails to satisfy the following three issues: (i) His ranking is contradicted with previous existing results, (ii) His advantages for repeated items is no longer valid after we adjust hesitant fuzzy elements with the same length and (iii) His proof for the component-wise ordering being preserved by his new ranking method is redundant. Our derivations will help researchers realize questionable results of the novel lexicographical ordering proposed by Farhadinia (2016).

Keywords: *Lexicographical ordering, Multi-attribute decision making, Hesitant fuzzy element.*

1. Introduction

Zadeh [10] constructed fuzzy sets to create a new research topic and then there are more than twenty thousand papers that had been published which were related to fuzzy sets. There are many different generalizations that are with respect to fuzzy sets, for example, hesitant fuzzy sets (HFS) which was introduced Torra [6]. Recently, Torra [6] had been cited by five hundred papers to reveal that HFS attracts attractions from many researchers. The key issue for HFS is the hesitant fuzzy element (HFE). Up to now, many articles tried to decide the order of HFEs. We just list a few in the following: For

examples, Wang et al. [7], Xia and Xu [8], Xu and Xia [9], Farhadinia [1,2,4]. In this technical note, we will focus on Farhadinia[4] to present an improvement.

2. Discussion for Farhadinia [4]

We recall important issues in Farhadinia [4] for our later examination. Interested readers please consider the original paper of Farhadinia [4]. Farhadinia [4] assumed that for a hesitant fuzzy element (HFE) $h(x)$ for $x \in X$ with cardinal number $|h| = m$ is denoted as

$h = \left\{\gamma^{(1)}, \gamma^{(2)}, ..., \gamma^{(m)}\right\}$ to satisfy $\gamma^{(k+1)} \geq \gamma^{(k)}$ for $k = 0, ..., m-1$.

Xia and Xu [8] assumed the score function by arithmetic mean:

$$S_{AM}(h) = \frac{1}{m}\sum_{k=1}^{m}\gamma^{(k)}. \tag{1}$$

For a HFE, $h(x)$ as $h = \left\{\gamma^{(1)}, \gamma^{(2)}, ..., \gamma^{(m)}\right\}$, Liao et al. [5] defined the derivation function v_{lxx} as

$$v_{lxx}(h) = \left(\frac{1}{m}\sum_{i \neq j=1}^{m}\left(\gamma^{(i)} - \gamma^{(j)}\right)^2\right)^{1/2} \tag{2}$$

In Farhadinia [4], he mentioned that Equation (2) should be revised to

$$v_{lxx}(h) = \left(\frac{2}{m(m-1)}\sum_{i \neq j=1}^{m}\left(\gamma^{(i)} - \gamma^{(j)}\right)^2\right)^{1/2} \tag{3}$$

In Torra [6], he recalled that

$$h_1 \oplus h_2 = \bigcup_{\gamma_1 \in h_1, \gamma_2 \in h_2}\left\{\gamma_1 + \gamma_2 - \gamma_1\gamma_2\right\}. \tag{4}$$

Liao et al. [5] defined a comparison law for two HFEs as follows:

If $S_{AM}(h_1) > S_{AM}(h_2)$, then $h_1 > h_2$;

If $S_{AM}(h_1) = S_{AM}(h_2)$, $v_{lxx}(h_1) > v_{lxx}(h_2)$, then $h_2 > h_1$.

If $h_1 = \left\{\gamma_1^{(1)}, \gamma_1^{(2)}, ..., \gamma_1^{(m)}\right\}$ and $h_2 = \left\{\gamma_2^{(1)}, \gamma_2^{(2)}, ..., \gamma_2^{(m)}\right\}$ are two HFEs, Farhadinia[2] defined the component-wise ordering of HFEs as

$h_1 \leqq h_2$ if and only if $\gamma_1^{(i)} \leq \gamma_2^{(i)}$, for $1 \leq i \leq m$. (5)

Farhadinia [4] mentioned that the number of values in different HFEs may be different. As assumed in many contributions made to the theory of HFEs (see e.g. [1,2,8,9], Farhadinia [4] extended the HFE with fewer elements by repeating its maximum element until it has the same length with the other HFE.

Farhadinia [4] defined a new deviation for h,

$$v_\phi(h) = \sum_{i=1}^{m-1}\phi\left(\gamma^{(i+1)} - \gamma^{(i)}\right) \tag{6}$$

where $\phi: [0,1] \to [0,1]$ increases that is satisfying $\phi(0) = 0$.

Farhadinia [4] defined a lexicographic order for HFEs. Given an HFE h, the related ranking vector of h is expressed as $R(h)$ satisfying $R(h) = \left(S_{AM}(h), v_\phi(h)\right)$, where $S_{AM}(h)$ is assumed by Equation (1) and $v_\phi(h)$ is denoted by Equation (6). Farhadinia [4] assumed his HFE lexicographic order as follows

$$h_1 < h_2 \text{ if and only if } R(h_1) <_{lex} R(h_2), \tag{7}$$

We recall the Theorem 3.1 of Farhadinia [4] in the following.

Theorem 3.1

Let $h_1 = \left\{\gamma_1^{(1)}, \gamma_1^{(2)}, ..., \gamma_1^{(m)}\right\}$ and $h_2 = \left\{\gamma_2^{(1)}, \gamma_2^{(2)}, ..., \gamma_2^{(m)}\right\}$ are two HFEs with $\gamma_1^{(i)} \leq \gamma_2^{(i)}$ for $i = 1, 2, ..., m$. Then, we get $R(h_1) \leq_{lex} R(h_2)$.

We cite an outline for the proof of Theorem 3.1 from Farhadinia [4].

Farhadinia [4] derived that $S_{AM}(h_1) \leq S_{AM}(h_2)$ and then he divided the proof into two cases: (a) $S_{AM}(h_1) < S_{AM}(h_2)$ and (b) $S_{AM}(h_1) = S_{AM}(h_2)$.

For case (a), it yielded that $R(h_1) <_{lex} R(h_2)$.

For case (b), he presented a lengthy proof to show that $\gamma_1^{(i+1)} - \gamma_1^{(i)} \leq \gamma_2^{(i+1)} - \gamma_2^{(i)}$, for $i = 1, 2, ..., m-1$, and then he obtained that $\phi\left(\gamma_1^{(i+1)} - \gamma_1^{(i)}\right) \leq \phi\left(\gamma_2^{(i+1)} - \gamma_2^{(i)}\right)$ to derive that $v_\phi(h_1) \leq v_\phi(h_2)$ and

then he obtained that $R(h_1) \leq_{lex} R(h_2)$.

In the next section, we will provide a simple proof for case (b).

3. Our proposed challenges and revisions

In the following, we will provide a simple proof for case (b) of Theorem 3.1 of Farhadinia [4].

From $\gamma_1^{(i)} \leq \gamma_2^{(i)}$ for $i = 1, 2, \ldots, m$ and $S_{AM}(h_1) = S_{AM}(h_2)$, we imply that $\gamma_1^{(i)} = \gamma_2^{(i)}$ for $i = 1, 2, \ldots, m$. Consequently, $\upsilon_\phi(h_1) = \upsilon_\phi(h_2)$ and

$R(h_1) = R(h_2)$ to derive that $R(h_1) =_{lex} R(h_2)$.

Remark. The lengthy proof of Farhadinia [4] for $\gamma_1^{(i+1)} - \gamma_1^{(i)} \leq \gamma_2^{(i+1)} - \gamma_2^{(i)}$, for $i = 1, 2, \ldots, m-1$, is a true statement.

In fact, from our derivations of $\gamma_1^{(i)} = \gamma_2^{(i)}$ for $i = 1, 2, \ldots, m$, and then it is trivial that $\gamma_1^{(i+1)} - \gamma_1^{(i)} = \gamma_2^{(i+1)} - \gamma_2^{(i)}$, for $i = 1, 2, \ldots, m-1$ to yield that their

assertion of $\gamma_1^{(i+1)} - \gamma_1^{(i)} \leq \gamma_2^{(i+1)} - \gamma_2^{(i)}$, for $i = 1, 2, \ldots, m-1$ is valid.

However, we must point out that the proof in Farhadinia [4] to show that for $i = 1, 2, \ldots, m-1$, $\gamma_1^{(i+1)} - \gamma_1^{(i)} \leq \gamma_2^{(i+1)} - \gamma_2^{(i)}$, is unnecessary.

We must point out that the assertion of Farhadinia [4] of Equation (3) should be further revised as

$$v_{lxx}(h) = \left(\frac{1}{m(m-1)} \sum_{i \neq j=1}^{m} \left(\gamma^{(i)} - \gamma^{(j)} \right)^2 \right)^{1/2} \quad (8)$$

or

$$v_{lxx}(h) = \left(\frac{2}{m(m-1)} \sum_{1=i<j}^{m} \left(\gamma^{(i)} - \gamma^{(j)} \right)^2 \right)^{1/2}. \quad (9)$$

We can observe that those derivation functions of Equations (2-3, 9) are only different by a constant. For example,

$$\left(\frac{1}{m} \sum_{i \neq j=1}^{m} \left(\gamma^{(i)} - \gamma^{(j)} \right)^2 \right)^{1/2} = c_1 \left(\frac{2}{m(m-1)} \sum_{i \neq j=1}^{m} \left(\gamma^{(i)} - \gamma^{(j)} \right)^2 \right)^{1/2}, \quad (10)$$

$$\left(\frac{2}{m(m-1)} \sum_{i \neq j=1}^{m} \left(\gamma^{(i)} - \gamma^{(j)} \right)^2 \right)^{1/2} = c_2 \left(\frac{1}{m(m-1)} \sum_{i \neq j=1}^{m} \left(\gamma^{(i)} - \gamma^{(j)} \right)^2 \right)^{1/2}, \quad (11)$$

and

$$\left(\frac{1}{m(m-1)} \sum_{i \neq j=1}^{m} \left(\gamma^{(i)} - \gamma^{(j)} \right)^2 \right)^{1/2} = c_3 \left(\frac{2}{m(m-1)} \sum_{1=i<j}^{m} \left(\gamma^{(i)} - \gamma^{(j)} \right)^2 \right)^{1/2} \quad (12)$$

with $c_1 = \sqrt{\frac{m-1}{2}}$, $c_2 = \sqrt{2}$ and $c_3 = 1$ such that for a pair of two HFEs h_1 and h_2 with the same length n, if $S_{AM}(h_1) = S_{AM}(h_2)$, then by any derivation function of Equations (2-3, 9), then the same ordering will be obtained.

The purpose of Equations (8-9) is to normalize the

value to yield $v_{lxx}(h) \leq 1$.

4. Review of numerical examples of Farhadinia[4]

Farhadinia [4] provided two examples. For the first one with three HFEs:

$h_1 = \{0.1, 0.3, 0.3, 0.3, 0.5\}$, $h_2 = \{0.1, 0.3, 0.3, 0.5\}$ and $h_3 = \{0.1, 0.3, 0.5\}$. (13)

He derived that by Equation (1),

$$S_{AM}(h_1) = S_{AM}(h_2) = S_{AM}(h_3) = 0.3, \qquad (14)$$

and by Equation (8),

$$v_{lxx}(h_1) = \sqrt{\frac{3}{75}}, \quad v_{lxx}(h_2) = \sqrt{\frac{4}{75}}, \quad \text{and} \quad v_{lxx}(h_3) = \sqrt{\frac{6}{75}}, \qquad (15)$$

to imply that

$$h_1 > h_2 > h_3. \qquad (16)$$

Remark

The computations of Equation (6) are consistent with our assertions of (8) or (9) to demonstrate that our revision for Equation (3) is valid.

Farhadinia [4] used his approach of Equation (6) to yield that

$$\upsilon_\phi(h_1) = \upsilon_\phi(h_2) = \upsilon_\phi(h_3) = 2\phi(0.2) \qquad (17)$$

to imply that

$$R(h_1) =_{lex} R(h_2) =_{lex} R(h_3) \qquad (18)$$

Farhadinia [4] concluded that his approach has the advantage that is invariant with respect to multiple occurrences of 0.3.

For his second numerical example, there are three HFEs:

$$h_4 = \{0.1, 0.5\} \quad , \quad h_5 = \{0.1, 0.2, 0.4, 0.5\} \quad \text{and}$$

$$h_6 = \{0.1, 0.2, 0.3, 0.4, 0.5\}. \qquad (19)$$

With $\phi(t) = t^2$, Farhadinia [4] derived that

$$R(h_4) = (0.3, 0.16) >_{lex} R(h_5) = (0.3, 0.06) >_{lex} R(h_6) = (0.3, 0.04) \qquad (20)$$

to imply that

$$h_4 > h_5 > h_6. \qquad (21)$$

He also mentioned that if he extended the length of h_4 and h_5 by the optimistically by repeating their maximum element until they have the same length with h_6, then $\bar{h}_4 = \{0.1, 0.5, 0.5, 0.5, 0.5\}$ and

$\bar{h}_5 = \{0.1, 0.2, 0.4, 0.5, 0.5\}$ to yield

$$S_{AM}(\bar{h}_4) = 0.42 > S_{AM}(\bar{h}_5) = 0.34 > S_{AM}(h_6) = 0.3 \qquad (22)$$

to imply that

$$h_4 > h_5 > h_6 \qquad (23)$$

that is consistent with his result of Equation (21).

5. Our discussion for his numerical examples

We recall that $S_{AM}(h)$ is the arithmetic mean to represent the tendency of data and $\upsilon_{lxx}(h)$ measures the derivation among data, the smaller the better. It is the common approach. Our recall is consistent with the definition of Liao et al. [5].

However, the lexicographic order proposed by Farhadinia [4] assumed that $S_{AM}(h_1) = S_{AM}(h_2)$, $\upsilon_\phi(h_1) < \upsilon_\phi(h_2)$, then $h_1 < h_2$.

We construct three HFEs:

$$h_7 = \{0, 0.6\}, \quad h_8 = \{0.1, 0.5\} \quad \text{and} \quad h_9 = \{0.2, 0.4\}, (24)$$

to find that

$$S_{SM}(h_7) = S_{AM}(h_8) = S_{AM}(h_9) = 0.3 \qquad (25)$$

and

$$\upsilon_\phi(h_7) = 0.36 > \upsilon_\phi(h_8) = 0.16 > \upsilon_\phi(h_9) = 0.04 \qquad (26)$$

with $\phi(t) = t^2$, and then by the lexicographic order proposed by Farhadinia [4] to imply that

$$h_9 < h_8 < h_7. \qquad (27)$$

However, we can observe that data from h_9 is more accumulated around the mean 0.3 than h_8

and h_7. Hence, intuitively researcher should expect that

$$h_7 < h_8 < h_9. \tag{28}$$

If we compute

$$\upsilon_{lxx}(h_7) = 0.6 > \upsilon_{lxx}(h_8) = 0.4 > \upsilon_{lxx}(h_9) = 0.2. \tag{29}$$

By the comparison law of Liao et al. [5], then

$$h_7 < h_8 < h_9, \tag{30}$$

that is consistent with intuition of Equation (28).

From the above discussion, we can say that the lexicographic order proposed by Farhadinia [4] is questionable.

For his first example, if we follow his approach for HFEs h_4, h_5 and h_6 to extend h_2 and h_3 to the length of h_1, then

$$\overline{h}_2 = \{0.1, 0.3, 0.3, 0.5, 0.5\} \quad \text{and}$$

$$\overline{h}_3 = \{0.1, 0.3, 0.5, 0.5, 0.5\}, \tag{31}$$

to imply that

$$S_{SM}(h_1) = 0.3 < S_{AM}(\overline{h}_2) = 0.34 < S_{AM}(\overline{h}_3) = 0.38 \tag{32}$$

such that the claim of Farhadinia [4] the invariant property is no longer valid.

For his second numerical example, we observe h_4, h_5 and h_6 to find out that h_6 is more clustering around the mean 0.3 such that we will provide a revision for the lexicographic ordering proposed by Farhadinia [4] as follows,

If $S_{AM}(h_1) < S_{AM}(h_2)$, then $h_1 < h_2$.

If $S_{AM}(h_1) = S_{AM}(h_2)$, $\upsilon_\phi(h_1) > \upsilon_\phi(h_2)$, then $h_1 < h_2$.

Based on our revision, we recall the result of Equation (20) in the following,

$$S_{AM}(h_4) = S_{AM}(h_5) = S_{AM}(h_6) = 0.3 \tag{33}$$

and

$$\upsilon_\phi(h_4) = 0.16 > \upsilon_\phi(h_5) = 0.06 > \upsilon_\phi(h_6) = 0.04 \tag{34}$$

to imply that

$$h_4 < h_5 < h_6, \tag{35}$$

that is consistent with our intuition and the derivation of Liao et al. [5] as

$$\upsilon_{lxx}(h_4) = \frac{4}{10} > \qquad \upsilon_{lxx}(h_5) = \frac{\sqrt{35/6}}{10}$$

$$> \upsilon_{lxx}(h_6) = \frac{\sqrt{5}}{10}. \tag{36}$$

6. Conclusion

We find three doubtful results in Farhadinia [4] with respect to his lexicographic ordering. Hence, we can advise researchers pay attention to the lexicographic ordering when applying this questionable ordering in their researches.

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