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Some Analytical Aspects of Categories

Authors

Dhanjit Barman

Research Scholar

Deptt. of Mathematics, Gauhati University, Guwahati, Assam

Email: *dhanjitbarman2011@gmail.com*

Postal Address: C/O Principal, Jalukbari H. S. School, Satmile, Jalukbari, Ghy-14.

ABSTRACT

In this paper we discuss some analytical aspects of categories. Here we see that if F : C→ D is a covariant functor then image of F will not form a subcategory of D. We provide an example to show it. Also we try to find some results of the category of rings (Ring), the category of sets (Set), the category of groups (Gp) and category of topological spaces (Top). We define some functors between categories and discuss their properties.

Key words: *Category, functor, morphism, monomorphism, epimorphism,bimorphism, isomorphism , balanced, normalcategory.*

INTRODUCTION

Here we discuss some analytical aspects of categories. We try to find some properties of the category of rings (Ring), category of sets (Set) , category of topological spaces (Top) and category of groups (Gp). Also we find some functors between categories and their properties. If F : C→ D is a covariant functor then, in general , its image will not form a subcategory of D.

PRELIMINARIES

For notions of category theory we shall in general follow the notation and terminology of Popescu [6]. However, we do deviate somewhat.

For C a category and A, B objects of C, Mor(A, B) denotes the set of morphisms from A to B.

We will also follow Popescu [6] for the definition of Preadditive, Additive and Preabelian and abelian category.

For kernel and cokernel we follow MacLane[2].

We follow the definition of retraction and coretraction from Popescu and Pareigis[11].

We shall use the definition of Balanced category from Mitchel [3] Monomorphism from Schubert [5] and epimorphism and isomorphism from Pareigis [11].

MAIN RESULTS

Let us consider,

Rng = the category of rings , whose objects are rings and morphisms are ring homomorphismms.

Rngu: the category of rings with unity together with unital (identity preserving) ring homomorphisms.

DivRing: the category of division rings together with unital ring homomorphisms.

PROPERTIES

1. DivRng is full subcategory of Ringu
2. Rngu is not full subcategory of Rng as every ring homomorphism between rings with identity is not unital.
3. Both DivRng and Rngu are subcategory of Rng.

Proposition 1.1: In DivRng , every epimorphism is bijective morphism.

Proof: Let f be an epimorphism in DivRng. At first we will prove that in DivRng every morphism is monomorphism.

Since the ideals of a division ring are {0} and R. So every (unital) ring homomorphism in DivRng is injective.(by considering the kernel of a unital ring homomorphism of division rings)And so a monomorphism

Therefore f is both epimorphism and monomorphism.

Hence f is bijective morphism.

Proposition 1.2: In Rng not all monomorphisms are kernels.

Proof: Let R ∈ obRng be an object in Rng. Let H be a subring, but not an ideal of R.

Then' i' is not kernel. But 'i'is monomorphism .

Proposition 1.3: DivRng has no initial object, no final object, no zero object and no zero morphism.

Proof: Let R, R' be two division rings with different characteristics. Then Mor(R , R') = φ. So DivRng has no initial object, no final object, no zero object and no zero morphism.

Proposition 1.4: Rngu and DivRng are not abelian category.

Proof : since Rngu and DivRng donot have zero morphisms so they canot be additive categories. And hence both are not abelian categories.

Proposition 1.5: Rng is also not abelian.

Proof: Rng has zero morphisms . But then also it is not abelian as sum of two rings is not a ring.

Proposition 1.6: Every covariant functor preserves retraction and section.

Proof : Let F : C → D be a covariant functor.

Let f: A → B be a retraction, then there is a morphism g :B → A such that fog = 1_B.

Now F(fog) = f(1_B)

= > F(f) o F(g) = $1_{F(B)}$ [since F is covariant]

= >F(f) is retraction.

Let f: A → B be a section, then there is a morphism g : B → A such that gof = 1_A.

Now F(gof) = f(1_A)

= > F(g)o F(f) = $1_{F(A)}$ [since F is covariant]

= >F(f) is section.

]

Proposition 1.7: Every covariant functor F : Set → C preserves monomorphism and epimorphism.

Proof: Let f: A → B is monomorphism. We will show that f is injective.

Suppose to the contrary , f is not injective . So we have a≠a' such that f(a) = f(a').

We will show that functions g,h can be constructed such that fog = foh

Implies g ≠ h (i.e f is not monomrphism).

Let us consider the functions g,h : C → A such that fog = foh , where C = {a , a'}.

Let us define g(a) = a, g(a') = a' and

h(a) = h(a') = a.

thus we have f og = foh but g ≠ h

Hence f is not monomorphism

Therefore monomorphism implies injective in Set.

Next we will prove that injective in Set is section.

Let f : A →B be injective in Set. Let us define g : B →A such that for a fixed a∈ A

g(b) = a' if f(a') = b

= a if b ∈ B – f(A) .

Then gof (a') = g (f (a')) = g(b) = a' 1_A (a')

= > gof = 1_A .

Hence f is section.

B y **proposition 1.6** we have F(f) is section in C.

But every section is monomorphism.

Thus F(f) is monomorphism in C.

Similarly it can be proved that in Set '' f is Epimorphism iff f is retraction."

By **proposition 1.6** F(f) is retraction.

But every retraction in a category is an epimorphism.

Thus F(f) is an epimorphism in C.

Problem: Provide an example of functor which does not preserve monomorphisms.

Soluton: Let us consider the forgetful functor F : DivRng → Rng.

Here in DivRng, every morphism is monomorphism but the image under F in Ring may not be monomorphism.

Let us consider ------

Set = the category of sets together with mapping between them

Fin Set = the category of finite sets and together with maps between them.

Inj = the category of sets together with the injective maps between them.

Surj = The category of sets together with the surjective maps between them.

Bij = The category of sets together with the bijective maps between them.

PROPERTIES:

1. Fin Set is full subcategory of Set.
2. Inj, Surj and Bij are sub categories of Set.
3. Set is not an abelian category as it does not have zero object.
4. Similarly Fin Set, Inj , Surj, Bij are also not abelian categories.

Proposition 1.8: Prove that if F: C → D is a functor between categories with zero object, then the following conditions are equivalent:

a) F preserves constant morphisms; a morphism f : A → B is *constant* provided that for any pair A′ r⇉s A of morphisms we have f o r = f o s.

b) F preserves coconstant morphisms;

c) F preserves zero morphisms;

d) F preserves zero objects.

Proof: (d) = > (c)

Let F preserves zero objects i.e. if Z is a zero object in C then F(Z) is zero object in D. Let g : A → B be a zero morphism in C.

Then either A or B or both A and B are zero objects. Thus by our assumption either F(A) or F(B) or both are zero objects in D.

Hence F(g) is zero morphism in D.

(c) = > (b), (a)

Let F preserves zero morphisms.

By definition of zero morphism , a zero morphism is both constant and coconstant morphism.

Hence F preserves both constant and coconstant morphisms.

(a) = > (d)

Let F preserves constant morphisms.

Let Z be a zero object in C. Let g : Z → B be a zero morphism.

Thus g is both constant and co constant morphisms, by definition.

Hence F(g); F(Z) → F(B) is both constant and coconstant morphisms in D.

Thus F(Z) is zero object in D.

(b) = > (d)

Let F preserves coconstant morphisms.

Let Z be a zero object in C. Let g : Z → B be a zero morphism.

Thus g is both constant and coconstant morphisms, by definition.

Hence F(g); F(Z) → F(B) is both constant and coconstant morphisms in D.

Thus F(Z) is zero object in D.

In the following every category is taken to be a full subcategory of, Ab, the category of abelian groups.

a) The category of torsion abelian groups (Tor) is an abelian category.

b) The category of torsion-free abelian groups (Torfree) is not an abelian category.

c) The category of fnitely generated abelian (FG-Ab) groups is an abelian category.

d) The category of divisible groups(Div) is not an abelian category.

Proposition 1.9: T: Ab → Tor is a covariant functor, where T sends every to its torsion subgroup and every group homomorphism to its restrictuion to the torsion subgroup.

Proof: Here T (A) = A', A' being torsion subgroup of A.

For f ∈ Mor (A, B), g ∈ Mor (C, A), T (f) ∈ Mor (A' , B') and T (g) ∈ Mor (C' , A') T(f) , T(g) are the restriction to the torsion subgroup such that T(f)(a')= a' and T(g)(c') = c'

Then I) T(fog) (c')= c' and {T(f)o T(g)}(c')= c'

Thus T(fog) = T(f) o T(g).

Similarly it can be shown that

ii) $T(1_A) = 1_{T(A)}$.

Hence T is a covariant functor.

Proposition 1.10: T: Ab → Torfree is a covariant functor, where T sends every group to its quotient by its torsion subgroup and every group homomorphism to the induced homomorphism.

Proof : It is obvious from the above proposition 1.9

Let us consider the following:

Top = the category of topological spaces equipped with continuous map between topological spaces.

Haus: the category of topological spaces with hausdorff property.

FinHaus: the category of finite topological spaces with hausdorff property.

ConnTop: the category of connected topological spaces.

DiscnTop: the category of disconnected topological spaces.

CompTop: the category of compact topological spaces.

DsTop: the category of discrete topologies.

IndsTop: The category of indiscrete topologies.

T_0 –Top: the category of T_0 topological spaces.

T_1 – Top: The category of T_1 spaces.

Proposition: The monomorphisms in Top are the injective continuous maps.

Proposition: The epimorphisms are the surjective continuous maps.

Proposition: The isomorphisms are the homeomorphisms.

PROPERTIES:

1. ConnTop, DiscnTop, CompTop ,DsTop, IndsTop, T_0 –Top , T_1 – Top are subcategories of Top.
2. DsTop, IndsTop is a subcategory of Haus.
3. IndsTop is a subcategory of CompTop.
4. IndsTop is a full subcategory of ConnTop.
5. DsTop is a full subcategory of DiscnTop.
6. Haus is a full subcategory of Top.
7. T_1 – Top is a subcategory of T0 –Top.
8. DsTop is a subcategory of T_0 –Top. But IndsTop is not a subcategory of T_0 –Top.
9. Haus is a subcategory of T_1 – Top. But the converse is not true.
10. FinHaus is a subcategory of DsTop.

Note: Φ is the initial object, and singletons (sets with exactly one elements) are final objects in Top. Top has no zero object nor zero *morphisms.* The same is true in Haus.

Proposition 1.11: Top is neither abelian nor normal nor conormal.

Proof: Since the very definitions of kernels and cokernels in a category already need the existence of (at least) zero morphisms, there are no kernels nor cokernels in Top. Thus, Top is neither abelian nor normal nor conormal.

Note: The same is true in Haus.

Note : The map U : Top → Set to the category of sets which assigns to each topological space the underlying set and to each continuous map the underlying function is forgetful functor.

Proposition 1.12: Let us consider the map T : Set→ Top which equips a given set with the discrete topology. And let us suppose the map I : Set → Top which equips a given set with the indiscrete topology. The both are functors and both of these functors are, in fact, right inverses to U i.e. U o T and U o I are equals to the identity functor on the category Set.

Proof: Let us consider the map T: Set→ Top such that for A, B ∈ ob Set and

f ∈ Mor(A, B)

T (A) = {A, D} where D is the collection of all the subsets of A.

And $T(f) = f_D$, f_D being continuous map between the discrere topology on A and on B,as a function between discrete topologies is continuous function.

Then

i) $T(fog) = (fog)_D$

$= f_D \text{ o } g_D$

$= T(f) \text{ o } T(g).$

ii) $T(1_A) = 1_{T(A)}.$

thus T is a covariant functor.

Similarly , it can be proved that I : Set → Top is also covariant functor.

Now (U o T) (A) = U (T(A)) = U ({A, D}) = A (by definition) = Id_{Set} (A)
(UoT) (f) = U (T(f)) = U(f_D) = f (by definition) = Id_{Set}(f)
Hence U oT = Id_{Set} .

Similarly it can be proved that U o I = Id_{Set}.

Proposition 1.13: The map T : Set→ Top which equips a given set with the discrete topology. and the map I : Set → Top which equips a given set with the indiscrete topology are full embeddings.

Proof : Let A , B ∈ obSet be any two objects and let f∈Mor(I(A) ,I(B)) be any morphism in Top. Then f is a continuous map as any function between indiscrete topologies is continuous map. Thus we have f in Set such that
I(f) = f. Therefore I is full.
Next let f,g∈ Mor (A, B) such that
I(f) = I(g)= > f = g
Hence I is embedding.
Similarly it can be proved that T is full embedding as any function between discrete topologies is continuous.

Theorem 1: (This theorem is highlighted by Gitalee Das , Gauhati University) If $(A,+)$ be an abelian group such that it is a direct sum $\oplus Q \oplus C(p_j^{k_j})$ of a number of copies of the additive group of rational numbers and a number of copies of additive finite cyclic groups where $p_j^{k_j} \mid m$, m is an integer , then there exists a Goldie ring R such that R^+ = A .

Proposition 1.14: Let Ab(add) be a category of abelian groups satisfying the condition of theorem1 and let GRng be the category of goldie rings. Then we have a covariant functor F : Ab(add) → GRng which sends every abelian group G in Ab (add) to R^+ of R in GRng and abelian group homomorphism to abelian group homomorphism.

Proof: Thee proof is obvious from the above theorem 1.

Theorem 2: (This theorem is highlighted by Gitalee Das , Gauhati University) If $(A,+)$ be an abelian group such that it is a direct sum $T \oplus C$ where T is a bounded group and C is a torsion free group admitting a Goldie ring structure with 1, then there exists a Goldie ring R such that $R^+ = A$.

Proposition1.15: Let Ab (bn , torfree) be the category of abelian groups which satisfy the condition of theorem2 and GRng be the category of Goldie rings. Then we can have a covariant functor G : Ab(bn , torfree) → GRng which sends every abelian group G in Ab (bn , torfree) to R^+ of R in GRng (by above theorem) and abelian group homomorphism to abelian group homomorphism.

Proof: It is obvious from the above theorem 2.

Theorem 3: (This theorem is highlighted by Gitalee Das , Gauhati University) If R satisfies a.c.c. on annihilators of subsets of M as a right R-module , then R^+ is finitely generated.

Proposition 1.16 : Let RngAcc be the category of all such ring R satisfying ascending chain condition (a.c.c.) on annihilators of subsets of M as a right R- module and FG-Ab be the category of finitely generated abelian groups, then we have a forgetful functor T : RngAcc → FG-Ab which sends each R in RngAcc to R^+ in FG-Ab (forgets the multiplication) and each ring homomorphism in RngAcc to group homomorphism in FG-Ab.

Proof: It is to prove, by using the above theorem 3.

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REFERENCES

1. Anderson,Frank W.& Fuller,Kent R., *Rings and Categories of Modules*, Springer- Verlag New York berlin Heidelberg London paris Tokyo Hong Kong Barcelona Budapast.
2. Mac Lane, S.,1971: *Categories for the Working Mathematician*, Springer-Verlag New York Berlin
3. Mitchel, Barry.1965: *Theory of Categories*, Academic Press New York and London.
4. Krishnan,V.S.,1981: *An introduction to Category Theory*, North Holland New York Oxford.
5. Schubert, Horst,1972: *Categories*, Springer-Verlag Berlin Heidelberg New York.
6. Popescu,N.,1973: *Abelian Categories with Applications to Rings and Modules*, Academic Press, London & New York.
7. Awodey, Steve.,2006: *Category Theory, Second Edition,*Clarendon Press,Oxford.
8. Borceux, Francis.,1994: *Hand Book of Categorical Algebra*, Cambridge University Press
9. Simmons, Harold.,2011:*An Introduction to Category Theory* ,Cambridge University Press.
10. Freyd, P.,1965: *Abelian Categories, An Introduction totheTheory of Functors*, A Harper International Edition, 0 jointly published by Harper & Row, NewYork, Evaston & London and JOHN WEATHERHILL INC. TOKYO.
11. Pareigis,Bodo.,1970:*Categories and Functors*, Academic Press New York, London.