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Homomorphisms on S-Valued Graphs

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ABSTRACT

Let $G = (V, E)$ be a simple graph with n vertices and m edges. In [6], we introduced the notion of S −valued graph G^S where S stands for the semiring. In this paper, we study the concept of homomorphism and isomorphism between two S −valued graphs and discuss some simple properties.

Keywords: *Semiring, finite graph, graph isomorphism, isomorphism of S − valued graphs.*

AMS Classification: 05C25, 16Y60, 20D45.

1. INTRODUCTION

Algebraic graph theory [3] can be viewed as an extension of graph theory in which algebraic methods are applied to problems about graphs. Jonathan S. Golan [4] has introduced the notion of S −valued graph where he considered a function $g: V \times V \to S$ such that $g(v_1, v_2) \neq \phi$. But nothing more has been dealt.

Graph representations are widely used for dealing with the structural information, in different domains such as Networks, Psycho-Sociology, Pattern Recognition etc. One important problem to be solved using such representation is the matching of two graphs or colourings in graphs. In order to achieve a good correspondence between two graphs, the most used concept is the one of graph isomorphism. However, in most of the cases, the bijective condition is too strong in the study of structure of graphs. Hence the concept of graph isomorphism has to be replaced by the most generic concept of graph homomorphism [2].

The notion of graph homomorphism introduces an equivalence relation in the class of graphs, thus forming equivalence classes of graphs which can be used to study the colouring or matching problems of a graph. In [6], we have introduced the notion of semiring valued graphs (simply called S −valued graphs). In [5] and [7], we have discussed the regularity and the degree regularity conditions on S −valued graphs. In this paper, we introduce the notion of homomorphisms and isomorphisms of S −valued graphs, we study whether the isomorphism of graphs preserves the regularity conditions or not.

2. PRELIMINARIES

In this section, we recall some basic definitions from the theory of semirings, graphs and S −valued graphs that are needed in sequel.

Definition 2.1. [4] Let S_1 and S_2 be semirings. A function $\beta: S_1 \to S_2$ is a homomorphism of semirings if

$$\beta(0_{S_1}) = 0_{S_2}$$

$\beta(a + b) = \beta(a) + \beta(b)$ and $\beta(a \cdot b) = \beta(a) \cdot \beta(b)\ \forall\ a, b \in S_1$.

Remark 2.2.

If $(S, +, \cdot)$ is a semiring with unit element, then the definition of homomorphism on semiring preserves $\beta(1_{S_1}) = 1_{S_2}$.

A homomorphism of semirings which is both injective and surjective is called an isomorphism. If there exist an isomorphism between semirings S_1 and S_2, we write $S_1 \cong S_2$.

If $\beta: S_1 \to S_2$ is a homomorphism of semirings then, $im(\beta) = \{\beta(a) | a \in S_1\}$ is a subsemiring of S_2.

Example 2.3. Let $S_1 = (Z^+ \cup \{0\}, +, \cdot)$ and

$$S_2 = (M_2(S_1), +, \cdot) = \left\{ A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid a, b, c, d \in Z^+ \cup \{0\} \right\}$$ be given two semirings.

Now define $\beta: S_1 \to S_2$ by $n \mapsto \begin{bmatrix} n & 0 \\ 0 & n \end{bmatrix}$ $n \in S_1$.

1. $0 \in S_1$ and $\beta(0) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in S_2$ is the zero element $\Rightarrow \beta(0_{S_1}) = 0_{S_2}$.
2. $1 \in S_1$ and $\beta(1) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in S_2$ is the zero element $\Rightarrow \beta(1_{S_1}) = 1_{S_2}$.
3. Let $m, n \in S_1$. Then $\beta(m+n) = \begin{bmatrix} m+n & 0 \\ 0 & m+n \end{bmatrix} = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} + \begin{bmatrix} n & 0 \\ 0 & n \end{bmatrix}$
   a. $= \beta(m) + \beta(n)$
4. $\beta(mn) = \begin{bmatrix} mn & 0 \\ 0 & mn \end{bmatrix} = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \cdot \begin{bmatrix} n & 0 \\ 0 & n \end{bmatrix} = \beta(m) \cdot \beta(n)$.

$\Rightarrow \beta$ is a semiring homomorphism.

Let $\beta(m) = \beta(n)$ for some $m, n \in S_1$. $\Rightarrow \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} = \begin{bmatrix} n & 0 \\ 0 & n \end{bmatrix} \Rightarrow m = n$.

Therefore β is 1-1. Clearly β is not onto.

Therefore β is a 1-1 semiring homomorphism from S_1 to S_2 but not onto.

Example 2.4. Let $S_1 = (\{0, a, b\}, +, \cdot)$ be a semiring and the binary operation '+' and ' $\cdot$ 'are given in the following Cayley tables:

+	0	a	b
0	0	a	b
a	a	a	b
b	b	b	b

·	0	a	b
0	0	0	0
a	0	a	b
b	0	b	b

Let $S_2 = (\{0, f, g, h\}, +, \cdot)$ be a semiring whose binary operation '+' and ' $\cdot$ 'are defined as in the following Cayley Tables:

+	0	f	g	h
0	0	f	g	h
f	f	f	h	h
g	g	h	h	h
h	h	h	h	h

·	0	f	g	h
0	0	0	0	0
f	0	f	g	h
g	0	g	h	h
h	0	h	h	h

Define $\beta: S_1 \to S_2$ by $\beta(0) = 0_{S_2}$ $\beta(a) = f$; $\beta(b) = h$.

Clearly $\beta(0_{S_1}) = 0_{S_2}$.

The multiplicative identity element in S_1 and S_2 are a and f respectively.

Clearly $a \mapsto f$.

That is $\beta(1_{S_1}) = 1_{S_2}$

$\beta(0+0) = 0 = \beta(0) + \beta(0)$, $\beta(a+b) = h = \beta(a) + \beta(b)$

$\beta(0+a) = f = \beta(0) + \beta(a)$, $\beta(b+0) = h = \beta(b) + \beta(0)$

$\beta(0+b) = h = \beta(0) + \beta(b)$, $\beta(b+a) = h = \beta(b) + \beta(a)$

$\beta(a+a) = f = \beta(a) + \beta(a)$, $\beta(b+b) = h = \beta(b) + \beta(b)$

$\beta(a+0) = f = \beta(a) + \beta(0)$

Therefore $\beta(a+b) = \beta(a) + \beta(b)$ for any $a, b \in S_1$.

$\beta(0 \cdot 0) = 0_2 = \beta(0) \cdot \beta(0)$; $\beta(a \cdot b) = h = \beta(a) \cdot \beta(b)$

$\beta(0\cdot a)=0_2=\beta(0)\cdot\beta(a);\ \beta(b\cdot 0)=0_2=\beta(b)\cdot\beta(0)$
$\beta(0\cdot b)=0_2=\beta(0)\cdot\beta(b);\ \beta(b\cdot a)=h=\beta(b)\cdot\beta(a)$
$\beta(a\cdot 0)=0_2=\beta(a)\cdot\beta(0);\ \beta(b\cdot b)=h=\beta(b)\cdot\beta(b)$
$\beta(a\cdot a)=f=\beta(a)\cdot\beta(a);\ \beta(a\cdot b)=\beta(a)\cdot\beta(b)\ \forall\, a,b\in S_1.$
Therefore β is a 1-1 semiring homomorphism but not onto.

Example 2.5. Let $S_1=(Z^+\cup\{0\},+,\cdot)$ and

$S_2=(M_2(S_1),+,\cdot)=\left\{A=\begin{bmatrix}0 & a\\ 0 & a\end{bmatrix} \mid a\in Z^+\cup\{0\}\right\}$ be given two semirings.

Define $\beta: S_1\to S_2$ by $a\mapsto\begin{bmatrix}0 & a\\ 0 & a\end{bmatrix}$ $a\in S_1$.

1. $0\in S_1$ and $\beta(0)=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}\in S_2$ is the zero element $\Rightarrow\beta(0_{S_1})=0_{S_2}$.
2. $1\in S_1$ and $\beta(1)=\begin{bmatrix}0 & 1\\ 0 & 1\end{bmatrix}\in S_2$ is the zero element $\Rightarrow\beta(1_{S_1})=1_{S_2}$.
3. Let $m,n\in S_1$. Then $\beta(m+n)=\begin{bmatrix}0 & m+n\\ 0 & m+n\end{bmatrix}=\begin{bmatrix}0 & m\\ 0 & m\end{bmatrix}+\begin{bmatrix}0 & n\\ 0 & n\end{bmatrix}$
   a. $=\beta(m)+\beta(n)$
4. $\beta(mn)=\begin{bmatrix}0 & mn\\ 0 & mn\end{bmatrix}=\begin{bmatrix}0 & m\\ 0 & m\end{bmatrix}\cdot\begin{bmatrix}0 & n\\ 0 & n\end{bmatrix}=\beta(m)\cdot\beta(n)$.

$\Rightarrow\beta$ is a semiring homomorphism.

Let $\beta(m)=\beta(n)$ for some $m,n\in S_1.\Rightarrow\begin{bmatrix}0 & m\\ 0 & m\end{bmatrix}=\begin{bmatrix}0 & n\\ 0 & n\end{bmatrix}\Rightarrow m=n$.

Therefore β is 1-1.

To every $A=\begin{bmatrix}0 & a\\ 0 & a\end{bmatrix}$ in $M_2(S_1)$ there exist $a\in Z^+$ such that $\beta(a)=A$.

$\Rightarrow\beta$ is onto.

Therefore β is a semiring isomorphism from S_1 to S_2.

Definition 2.6. [1] A graph G is an ordered triplet $(V(G),E(G),\psi_G)$ consisting of a non empty set $V(G)$ of vertices, a set $E(G)$, disjoint from $V(G)$, of edges and an incidence function ψ_G that associates with each edge of G an unordered pair of (not necessarily distinct) vertices of G.

Definition 2.7 [1] The degree of a vertex in a graph is defined to be the number of edges incident with that vertex. A graph in which all vertices are of equal degree is called a regular graph.

Definition 2.8. [1] Two graphs G and G' are said to be isomorphic (written $\cong G'$) if there are bijections $\theta: V(G)\to V(G')$ and $\phi: E(G)\to E(G')$ such that $\psi_G(e)=uv$ iff $\psi'_G(\phi(2))=\theta(u)\theta(v)$; such a pair (θ,ϕ) of mappings is called an isomorphism between G and G'.

Now, we recall the definition of $S-$valued graphs.

Definition 2.9. [6] Let $G=(V,E\subset V\times V)$ be the underlying graph with $V,E\neq\phi$. For any semiring $(S,+,\cdot)$, a Semiring-valued graph (or a S-valued graph) G^S is defined to be the graph $G^S=(V,E,\sigma,\psi,$ where $\sigma: V\to S$ and $\psi: E\to S$ is defined to be

$$\psi(x,y)=\begin{cases}\min\{\sigma(x),\sigma(y)\} & if\quad \sigma(x)\preccurlyeq\sigma(y)\ or\ \sigma(y)\preccurlyeq\ \sigma(x)\\ 0 & otherwise\end{cases}$$

for every unordered pair (x,y) of $E\subset V\times V$. We call σ, a $S-$vertex set and ψ, a $S-$edge set of $S-$valued graph G^S.

Definition 2.10. [6] If $\sigma(x)=a,\ \forall\, x\in V$ and some $a\in S$ then the corresponding $S-$valued graph G^S is called a vertex regular $S-$valued graph (or simply vertex regular).

Definition 2.11. [6] A $S-$valued graph G^S is said to be an edge regular $S-$valued graph (or simply edge regular) if $\psi(x,y)=a$ for every $(x,y)\in E$ and for some $a\in S$.

Definition 2.12. [6] A S −valued graph G^S is said to be S −regular if it is both a vertex regular and edge regular S −valued graph.

Definition 2.13. [5] Let G^Sbe a S −valued graph corresponding to an underlying graph G, and $a \in S$. G^S is said to be a (a,k) − regular if it satisfies the following conditions:

The crisp graph G is k −regular.

$\sigma(v) = a$ for every $v \in V$.

Definition 2.14. [7] Let $G = (V, E)$ be the given crisp graph with n vertices and m edges. The Order of a S −valued graph G^S is defined as

$$p_S = (\sum_{v \in V} \sigma(v), n)$$

where n is order of the underlying graph G.

Definition 2.15. [7] Let $G = (V, E)$ be the given crisp graph with n vertices and m edges. The Size of a S −valued graph G^S is defined as

$$q_S = (\sum_{(u,v) \in E} \psi_{(u,v)}, m)$$

where m is the size of the underlying graph G.

Definition 2.16 [7] The degree of the vertex v_i of the S −valued graph G^S is defined as

$$\deg_S(v_i) = (\sum_{(v_i, v_j) \in E} \psi_{(v_i, v_j)}, l)$$

where l is the number of edges incident with v_i.

Definition 2.17. [7] A S −valued graph G^S is said to be d_s −regular if for every $v \in V, \deg_S(v) = (a, n)$ for some $a \in S$ and $n \in \mathbb{Z}^+$.

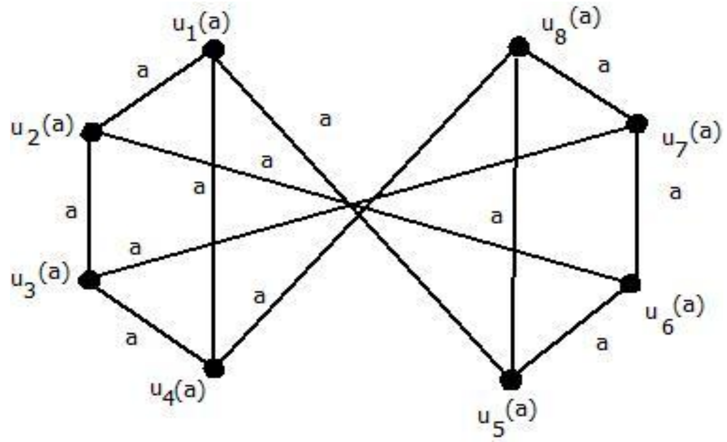
Example 2.18. Let $S = (\{0, a, b\}, +, \cdot)$ be a semiring with the binary operation '+' and ' · ' are given in the following Cayley tables:

+	0	a	b
0	0	a	b
a	a	a	b
b	b	b	b

·	0	a	b
0	0	0	0
a	0	a	B
b	0	b	B

Clearly, S is both multiplicatively and additively commutative and hence it is a commutative semiring. Let G_1^S be a S −valued graph given below.

G_1^S:



In G_1^S, we have $p_S = (a, 8)$ $q_S(a, 12)$ $\deg_S(u_1) = (a, 3)$. In particular, G_1^S is a d_s −regular graph.

3. HOMOMORPHISMS ON S-VALUED GRAPHS

In this section, we introduce the notion of homomorphism of S −valued graphs and study some simple properties satisfied by homomorphic classes of S −valued graphs.

Definition 3.1. Let $G_1^{S_1} = (V_1, E_1, \sigma_1, \psi_1)$ and $G_2^{S_2} = (V_2, E_2, \sigma_2, \psi_2)$ be S_1 −valued and S_2 −valued graphs respectively. A mapping $\phi = (\alpha, \beta): G_1^{S_1} \to G_2^{S_2}$ is called a S −valued homomorphism if

$\alpha: V_1 \to V_2$ is a graph isomorphism.

$\beta: S_1 \to S_2$ is a semiring homomorphism such that $\beta(\sigma_1(u_i)) = \sigma_2(\alpha(u_i))$ $\forall\, u_i \in V_1$ and $\beta\left(\psi_1(u_i, u_j)\right) = \psi_2\left(\alpha(u_i), \alpha(u_j)\right)$ $\forall\, (u_i, u_j) \in E_1$.

Remark 3.2.

1. If $S_1 = S_2 = S$ and $\beta = I: S \to S$ then $\phi(\alpha, I) = \alpha$. (ie) ϕ coincides with graph isomorphism.
2. If $G_1 = G_2$ and $\alpha = I: V_1 \to V_2$ then $\phi(I, \beta) = \beta$. (ie) ϕ coincides with semiring homomorphism.
3. If $\phi = (I_V, I_S)$, then it is a trivial automorphism on S −valued graphs.

Example 3.3. Let $S_1 = (\{0, a, b\}, +, \cdot)$ be a semiring with the binary operation '+' and $\cdot$ are given in the following Cayley tables:

+	0	a	b
0	0	a	b
a	a	a	b
b	b	b	b

·	0	a	b
0	0	0	0
a	0	a	b
b	0	b	b

Clearly, S_1 is both multiplicatively and additively commutative and hence it is a commutative semiring.

(ie) 0 is the additive identity element

a is the multiplicative identity element.

and a and b are both additively and multiplicatively idempotent elements.

Let $S_2 = (\{0, f, g, h\}, +, \cdot)$ be a semiring whose binary operators '+' and $\cdot$ are defined as in the following Cayley Tables:

+	0	f	g	h
0	0	f	g	h
f	f	f	h	h
g	g	h	h	h
h	h	h	h	h

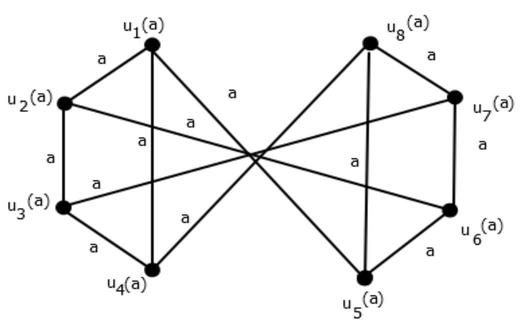
·	0	f	g	h
0	0	0	0	0
f	0	f	g	h
g	0	g	h	h
h	0	h	h	h

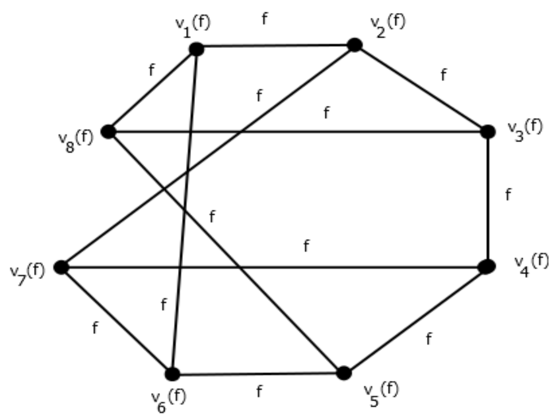
Clearly f is an unit element in S_2.

Define $\beta: S_1 \to S_2$ by $0_1 \mapsto 0_2$; $a \mapsto f$; $b \mapsto h$.

Then β is a semiring homomorphism which is 1-1 but not onto.

Let $G_1^{S_1}$ and $G_2^{S_2}$ be given as follows:





Define $\alpha: G_1 \to G_2$ by

$u_1 \mapsto v_3,\ u_2 \mapsto v_8,\ u_3 \mapsto v_5,\ u_4 \mapsto v_4,\ u_5 \mapsto v_2,\ u_6 \mapsto v_1,\ u_7 \mapsto v_6,\ u_8 \mapsto v_7$

$(u_1,u_2) \mapsto (v_3,v_8),\ (u_2,u_3) \mapsto (v_8,v_5),\ (u_3,u_4) \mapsto (v_5,v_4),\ (u_4,u_1) \mapsto (v_4,v_3),$

$(u_5,u_6) \mapsto (v_2,v_1),\ (u_6,u_7) \mapsto (v_1,v_6),\ (u_7,u_8) \mapsto (v_6,v_7),\ (u_8,u_5) \mapsto (v_7,v_2),$

$(u_5,u_1) \mapsto (v_2,v_3),\ (u_6,u_2) \mapsto (v_1,v_8), (u_7,u_3) \mapsto (v_6,v_5), (u_8,u_4) \mapsto (v_7,v_4)$

Then α is a graph isomorphism.

Since $\sigma_1(u_i) = a\ \forall\ i = 1,2,\cdots,8,\ \beta(\sigma_1(u_i)) = \beta(a) = f = \sigma_2(\alpha(u_i))\ \forall\ i.$

Clearly $\beta\left(\psi_1(u_i,u_j)\right) = \beta(a) = f = \psi_2\left(\alpha(u_i),\alpha(u_j)\right)\ \forall\ (u_i,u_j) \in E_1.$

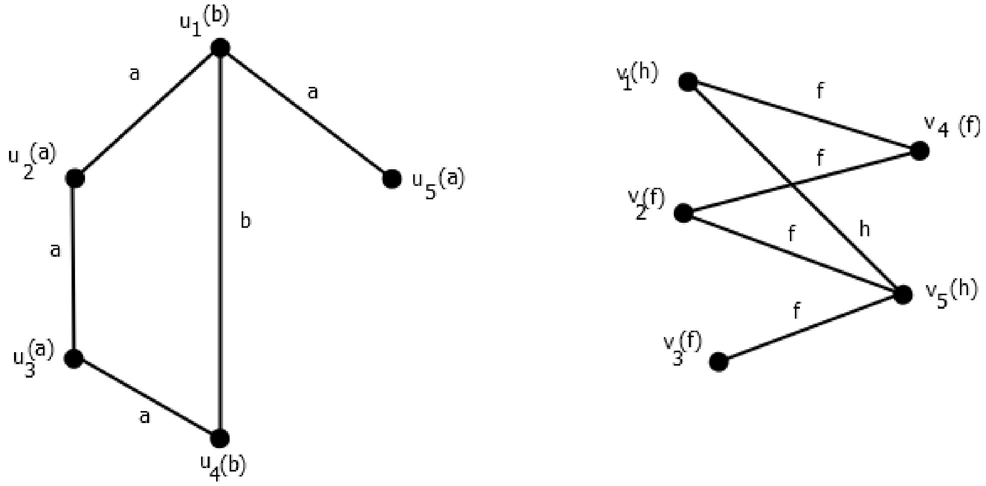
Therefore $\phi = (\alpha,\beta): G_1^{S_1} \to G_2^{S_2}$ is a S −valued homomorphism and $\phi(G_1^{S_1})$ is a $(f,3)$ −regular S −valued graphs.

That is $(\beta(a),3)$ −regular S −valued graph.

Remark 3.4. In the above example, a S −valued homomorphism preserves vertex regularity and hence edge regularity. Further, if $G_1^{S_1}$ is a (a,k) −regular graph, then $\phi(G_1^{S_1})$ is a $(\beta(a),k)$ −regular graph.

Example 3.5. Consider a semiring homomorphism β from example 3.3.

Let $G_1^{S_1}$and $G_2^{S_2}$ be given as follows:



Define $\alpha: G_1 \to G_2$ by $u_1 \mapsto v_5,\ u_2 \mapsto v_2,\ u_3 \mapsto v_4,\ u_4 \mapsto v_1,\ u_5 \mapsto v_3$

$(u_1,u_2) \mapsto (v_5,v_2), (u_2,u_3) \mapsto (v_2,v_4), (u_3,u_4) \mapsto (v_4,v_1),$

$(u_4,u_1) \mapsto (v_1,v_5), (u_5,u_1) \mapsto (v_3,v_5)$

Then α is a graph isomorphism.

Clearly, $\beta(\sigma_1(u_i)) = \sigma_2(\alpha(u_i))\ \ \forall\, u_i \in E_1.$

And $\beta\left(\psi_1(u_i,u_j)\right) = \psi_2\left(\alpha(u_i),\alpha(u_j)\right)\ \forall\ (u_i,u_j) \in E_1.$

Therefore $\phi = (\alpha,\beta): G_1^{S_1} \to G_2^{S_2}$is a S −valued homomorphism.

Remark 3.6. In above example 3.5, $G_1^{S_1}$ and $\phi(G_1^{S_1})$ are not a vertex, edge , (a,k) and degree regular S −valued graphs.

Theorem 3.7. If $\phi = (\alpha,\beta)$ is a S −valued homomorphism from a vertex regular graph $G_1^{S_1}$ with S_1 −vertex set $\{a\}$ into a S_2 −valued graph $G_2^{S_2}$ then $\phi(G_1^{S_1})$ is a S_2 −vertex regular graph with S_2 −vertex set $\{\beta(a)\}$.

Proof: Let $\phi = (\alpha,\beta): G_1^{S_1} \to G_2^{S_2}$ be a S −valued homomorphism

where $G_1^{S_1} = (V_1,E_1,\sigma_1,\psi_1),\ G_2^{S_2} = (V_2,E_2,\sigma_2,\psi_2),\ \ O(V_1) = O(V_2),\ O(E_1) = O(E_2)$

$\beta(\sigma_1(u)) = \sigma_2(\alpha(u))\ \ \forall\, u \in V_1$ and $\beta\left(\psi_1(u_i\,,u_j)\right) = \psi_2\left(\alpha(u_i),\alpha(u_j)\right)\ \forall\ (u_i,u_j) \in E_1.$

Since $G_1^{S_1}$ is a vertex regular graph with S_1 −vertex set $\{a\}$, $\sigma_1(u) = a\ \ \forall\, u \in V_1$ and for some $a \in S_1$.

Let $v \in V_2$. Since α is a graph isomorphism, there exist a $u \in V_1$ such that $\alpha(u) = v$.
Therefore $\sigma_2(v) = \sigma_2\big(\alpha(u)\big) = \beta\big(\sigma_1(u)\big) = \beta(a)$.
Since v is arbitrary, $\sigma_2(v) = \beta(a)\ \forall\ v \in V_2$ and for some $\beta(a) \in S_2$.
Therefore $G_2^{S_2}$ is a S_2 −vertex regular graph with S_2 −vertex set $\{\beta(a)\}$.

Corollary 3.8. If $\phi = (\alpha, \beta)$ is a $S-$valued homomorphism from a S_1-regular graph $G_1^{S_1}$ into a S_2 −valued graph $G_2^{S_2}$ then $\phi(G_1^{S_1})$ is a S_2 −regular graph with S_2 −vertex set $\{\beta(a)\}$.

Proof: Since $G_1^{S_1}$ is regular, $G_1^{S_1}$ is a S_1 −vertex regular graph. By theorem 3.7, $\phi(G_1^{S_1})$ is a S_2 −vertex regular graph and hence it is a S_2 −edge regular graph with S_2 −vertes set $\{\beta(a)\}$.
Therefore $\phi\big(G_1^{S_1}\big)$ is both vertex and edge regular.
$\Rightarrow \phi(G_1^{S_1})$ is a S_2 −regular graph.

Theorem 3.9. If $\phi = (\alpha, \beta)$ is a $S-$valued homomorphism from a S_1-edge regular graph $G_1^{S_1}$ with S_1-edge set $\{a\}$ into a S_2-valued graph $G_2^{S_2}$ and if $\beta(a) = \beta\big(\sigma_1(u)\big)$, $\forall\, u \in V_1$ then $\phi\big(G_1^{S_1}\big)$ is a S_2 −edge regular graph .

Proof: Let $G_1^{S_1} = (V_1, E_1, \sigma_1, \psi_1)$ be a S_1 −edge regular graph.
Therefore $\psi_1\big(u_i, u_j\big) = a$ for some $a \in S_1$ and for all $\big(u_i, u_j\big) \in E_1$.
Let $G_2^{S_2} = (V_2, E_2, \sigma_2, \psi_2)$ be S_2 −valued graph.
Let $\phi = (\alpha, \beta): G_1^{S_1} \to G_2^{S_2}$ be a S −valued homomorphism.
Since α is a graph isomorphism, to every $\big(v_i, v_j\big) \in E_2$, there exist $\big(u_i, u_j\big) \in E_1$ such that $\big(v_i. v_j\big) = \Big(\alpha(u_i), \alpha\big(u_j\big)\Big) \in E_2$.

$$\begin{aligned}\text{Therefore } \psi_2\big(v_i, v_j\big) &= \psi_2\left(\alpha(u_i), \alpha\big(u_j\big)\right)\\ &= \min\left\{\sigma_2\big(\alpha(u_i)\big), \sigma_2\left(\alpha\big(u_j\big)\right)\right\}\\ &= \min\left\{\beta\big(\sigma_1(u_i)\big), \beta\left(\sigma_1\big(u_j\big)\right)\right\}\\ &= \beta(a) \qquad \left(\text{since } \beta(a) = \beta\big(\sigma_1(u_i)\big)\right)\ \forall\, i.\end{aligned}$$

Therefore $\psi_2\big(v_i, v_j\big) = \beta(a)\ \ \forall\, i, j$.
$\Rightarrow G_2^{S_2}$ is a S_2 −edge regular graph if $\beta(a) = \beta\big(\sigma_1(u_i)\big)\ \ \forall\, i$.

Remark 3.10. From the above theorem, in general S −valued homomorphism does not preserve S −edge regularity.

Theorem 3.11. If $\phi = (\alpha, \beta)$ is a $S-$valued homomorphism from a $(a, k)-$regular graph $G_1^{S_1}$ into a S_2 −valued graph $G_2^{S_2}$ then $\phi(G_1^{S_1})$ is a $(\beta(a), k\,)$ −regular graph.

Proof: Let $G_1^{S_1} = (V_1, E_1, \sigma_1, \psi_1)$ be a (a, k) −regular graph.
Therefore $\sigma_1(u) = a\ \forall\ u \in V_1$ and for some $a \in S_1$ and $\deg(u) = k\ \ \forall\ u \in V_1$.
Let $G_2^{S_2} = (V_2, E_2, \sigma_2, \psi_2)$ be a S_2 −valued graph such that $O(V_1) = O(V_2)$ and $O(E_1) = O(E_2)$.
Let $\phi = (\alpha, \beta): G_1^{S_1} \to G_2^{S_2}$ be a S −valued homomorphism.
Therefore $\alpha: V_1 \to V_2$ is a graph isomorphism.
$\Rightarrow$ It is a bijective edge preserving map and there will be an equal number of vertices having equal degree.
Therefore to every $u \in V$, there exist only one $v \in V_2$ such that $\deg(u) = \deg(v)$.
Since $\deg(u) = k\ \ \forall\ u \in V_1$ and $O(V_1) = O(V_2), \deg(v) = k\ \ \forall\ v \in V_2$ (1)
By theorem 3.7, S −valued homomorphism preserves vertex regularity with S_2 −vertex set $= \{\beta(a)\}$ and $\sigma_2(v) = \beta(a)\ \forall\ v \in V_2$. (2)

From (1) and (2), $G_2^{S_2}$ is a $(\beta(a), k)$ −regular graph.

Theorem 3.12. Let $\phi = (\alpha, \beta): G_1^{S_1} \to G_2^{S_2}$ be a S −valued homomorphism then

1. If $G_1^{S_1}$ is a S_1 −regular with S_1 −vertex set $\{a\}$, $a \in S_1$ then
   a) Order of $G_2^{S_2}$ is $p_{S_2} = (\sum_{v \in V_2} \beta(a), n)$, $n = O(V_2)$. Further if $\beta(a) \in S_2$ is additively idempotent then $p_{S_2} = (\beta(a), n)$.
   b) The Size of the S −valued graph $G_2^{S_2} = q_{S_2} = (\sum_{(v_i,v_j) \in E_2} \beta(a), m)$ where m is the number of edges in $G_2^{S_2}$. And if $\beta(a)$ is additively idempotent in S_2 then $q_{S_2} = (\beta(a), m)$.
2. If $u_i \in V_1$ such that $\deg_{S_1}(u_i) = (a, l)$, l is the number of edges incident with u_i then there is a vertex in $G_2^{S_2}$ with degree $(\beta(a), l)$ where $\beta(a)$ is additively idempotent.

If $G_1^{S_1}$ is a d_{S_1} −regular (degree regular S_1 −valued graph) with S_1 −vertex set $\{a\}$, and if $\beta(a)$ is additively idempotent then $\phi(G_1^{S_1})$ is a d_{S_2} −regular graph.

Proof:

1. Let $G_1^{S_1}$ is a S_1 −regular with S_1 −vertex set $\{a\}$.
   a) By corollary 3.8, $\phi(G_1^{S_1})$ is a S_2 −regular with S_2 −vertex set $\{\beta(a)\}$.
   Therefore $\sigma_2(v) = \beta(a)\ \forall\ v \in V_2$.
   Since α is a graph isomorphism, $O(V_1) = O(V_2) = n$ and to every $v \in V_2$ there exist $u \in V_1$ such that $v = \alpha(u)$.
   Therefore $O(G_2^{S_2}) = p_{S_2} = (\sum_{v \in V} \sigma_2(v), n)$
   1. $= (\sum_{v \in V_2} \sigma_2(\alpha(u)), n)$
   2. $= (\sum_{v \in V_2} \beta(\sigma_1(u)), n)$
   3. $= (\sum_{v \in V_2} \beta(a), n)$ $\quad (\because\ \sigma_1(u) = a\ \forall\ u \in V_1)$

   If $\beta(a)$ is an additive idempotent element in S_2, then $\beta(a) + \beta(a) = \beta(a)$.
   Therefore $p_{S_2} = (\beta(a), n)$.
   b) Since α is graph isomorphism, $O(E_1) = O(E_2) = m$.
   Size of $G_2^{S_2} = q_{S_2} = (\sum_{(v_i,v_j) \in E_2} \psi_2(v_i, v_j), m)$ (1).
   ($\because\ \alpha$ is onto there exist $u_i, u_j \in V_1$ such that $\alpha(u_i) = v_i, \alpha(u_j) = v_j$)
   Consider $\psi_2\left(\alpha(u_i), \alpha(u_j)\right) = \min\left\{\sigma_2(\alpha(u_i)), \sigma_2\left(\alpha(u_j)\right)\right\}$
   a. $= \min\left\{\beta(\sigma_1(u_i)), \beta\left(\sigma_1(u_j)\right)\right\}$
   b. $= \min\{\beta(a), \beta(a)\}$
   c. $= \beta(a)\ \forall\ i, j.$

   b. $\Rightarrow q_{S_2} = (\sum_{(v_i,v_j)} \beta(a), m)$

   Suppose $\beta(a) \in S_2$ is additively idempotent, then $\sum \beta(a) = \beta(a)$.
   Therefore $q_{S_2} = (\beta(a), m)$.
   Given $\deg_{S_1}(u_i) = (a, l)$, $l = \deg(u_i)$.
   Since α is a graph isomorphism, there exist $v_i \in V_2$ such that $\alpha(u_i) = v_j$ having degree l.
   Therefore $\deg_{S_2}(v_i) = (\sum_{(v_i,v_j) \in E_2} \psi(v_i, v_j), l)$
   $= (\sum_{(v_i,v_j) \in E_2} \beta(a), l)$
   Since $\beta(a) \in S_2$ is additively idempotent, $\deg_{S_2}(v_i) = (\beta(a), l)$.

2. Given $G_1^{S_1}$ is a d_{S_1} −regular graph.
   Therefore $\deg_{S_1}(u) = (a, n)\ \forall\ u \in V_1,\ and\ some\ a \in S,$

$$n = \deg(u)\,, \sigma(u) = a\ \forall u \in V_1.$$

   (ie), G_1 is a n −regular graph. Since α is a graph isomorphism, $\phi(G_1^{S_1})$ is also a n −regular graph.
   Therefore $\deg(v) = n\ \ \forall\ v \in V_2$.
   Let $v \in V_2$ be arbitrary. Then there exist $u \in V_1$ such that $\alpha(u) = v$.
   i. $\deg_{S_2}(v) = (\sum_{(v,v_i)\in E_2} \psi_2(v, v_i), n)$
      1. $= (\sum_{(v,v_i)\in E_2} \psi_2(\alpha(u), \alpha(u_i)), n\ )$
      2. $= (\sum_{(v,v_i)\in E_2} \beta(a), n)$
      3. $= (\beta(a), n)$ (since $\beta(a) + \beta(a) = \beta(a)$).
   Therefore $\deg_{S_2}(v) = (\beta(a), n)\ \forall\ v \in V_2$.
   $\Rightarrow \phi(G_1^{S_1})$ is a d_{S_2} −regular graph.

CONCLUSION

In our further paper, we are going to extend S −valued homomorphism into S −valued isomorphisms.

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